Cosmology: 2015

Solutions: week 6

Exercise 1: Stress - energy tensor

a) It is convenient to split the action in its Einstein-Hilbert part and matter sector:

$$S^{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g(x)} \; R$$

$$S^{matter} = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right)$$

Obviously, S^{EH} is independent of ϕ. Thus

$$\frac{\delta S^{EH}}{\delta \phi} = 0$$

Then

$$\frac{1}{\sqrt{-g(y)}} \frac{\delta S^{matter}}{\delta \phi(y)}$$

$$= \frac{1}{\sqrt{-g(y)}} \int d^4x \sqrt{-g(x)}$$

$$\Big\{ -\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \delta^4(x-y) \right) \partial_\nu \phi$$

$$- \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \delta^4(x-y))$$

$$- \frac{\partial V}{\partial \phi} \delta^4(x-y) \Big\}$$

The first two terms are combined, integrating by parts as always. we assume that the fields fall off at infinity, so that there are no surface - terms

$$= \frac{1}{\sqrt{-g(y)}} \int d^4x \sqrt{-g(x)} \cdot \delta^4(x-y)$$

$$\left\{ \frac{1}{\sqrt{-g(x)}} \partial_\mu \left(\sqrt{-g(x)}\; g^{\mu\nu} \partial_\nu \phi \right) - \frac{\partial V}{\partial \phi} \right\}$$

Carrying out the integral with the δ- function and cancelling the determinants yields the e.o.m. for the scalar field:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\; g^{\mu\nu} \partial_\nu \phi \right) - \frac{\partial V}{\partial \phi} = 0 \qquad (*)$$

$\square$

For $g_{\mu\nu} = \eta_{\mu\nu}$ we have:

$$\sqrt{-\eta} = 1 \quad , \quad \partial_\mu \eta_{\alpha\beta} = 0$$

and

$$\frac{\partial}{\partial \phi} \left(\frac{1}{2} m^2 \phi^2 \right) = m^2 \phi$$

note: no δ- distribution here, since by construction $\frac{\partial V}{\partial \phi}$ is not a variation with respect to a field.

Substitute into (*):

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 \phi = 0$$

use $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ to obtain:

$$(\partial_t^2 - \partial_x^2 + m^2)\, \phi = 0$$

b) We first compute the variation of the Einstein - Hilbert term

$$\delta S^{EH} = \frac{1}{16\pi G} \int d^4x \, \left(\delta\sqrt{-g}\, R + \sqrt{-g}\, \delta R \right)$$

use identities:

$$= \frac{1}{16\pi G} \int d^4x \, \{ R \left(-\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\right) + \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \dots)$$

$$= \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta g^{\mu\nu}$$

Thus

$$\frac{1}{\sqrt{-g}} \frac{\delta S^{EH}}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)$$

Per definition

$$T^{\phi}_{\mu\nu} \equiv - \frac{2}{\sqrt{-g}} \frac{\delta S^{matter}}{\delta g^{\mu\nu}}$$

Thus:

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0$$

$$\overset{!}{=} \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \underbrace{\frac{1}{\sqrt{-g}} \frac{\delta S^{matter}}{\delta g^{\mu\nu}}}_{= -\frac{1}{2} T^{\phi}_{\mu\nu}} = 0$$

Thus

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^{\phi}_{\mu\nu}$$

need to compute $T^{\phi}_{\mu\nu}$ explicitly:

$$\delta_g S^{matter} = \int d^4x \left[(\delta \sqrt{-g}) \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V \right) + \sqrt{-g} \left(-\frac{1}{2} (\delta g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi \right) \right.$$

$$= \int d^4x \sqrt{-g}\, \delta g^{\mu\nu} \left\{ -\frac{1}{2} g_{\mu\nu} \left(-\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V \right) - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \right\}$$

Then

$$T^{\phi}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V \right)$$

c) Evaluation for FRW metric

$$T^{\phi}_{00} = \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + V$$

$$T^{\phi}_{xx} = -a^2\left(-\frac{1}{2}\dot{\phi}^2 + V\right)$$

we have $u^{\alpha} = (1, 0, 0, 0)$ (The scalar is at rest w.r.t cosmic coordinates)

The comparison then yields:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V$$

$$p = \frac{1}{2}\dot{\phi}^2 - V$$

Exercise 2: Conservation of the stress-energy tensor

a) note:

$$\delta S^{\text{matter}} = \frac{1}{2} \int d^4x \sqrt{-g}\; T^{\mu\nu}\, \delta g_{\mu\nu}$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g}\; T_{\mu\nu}\, \delta g^{\mu\nu}$$

use definition of $T_{\mu\nu}$:

$$T_{\mu\nu} \overset{!}{=} -\frac{2}{\sqrt{-g}} \frac{\delta S^m}{\delta g^{\mu\nu}}$$

$$= -\frac{2}{\sqrt{-g}} \left(-\frac{1}{2}\right) \int d^4x \sqrt{-g}\; T_{\alpha\beta} \cdot \frac{1}{2} \left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu\right) \cdot \delta^4(x-y)$$

$$= \frac{1}{\sqrt{-g}} \int d^4x \sqrt{-g}\; T_{\mu\nu}\, \delta^4(x-y)$$

$$= T_{\mu\nu}(y)$$

b) We start with the left-hand-side:

$$D_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\mu\lambda} T^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu} T^\mu{}_\lambda$$

$$= \partial_\mu T^\mu{}_\nu$$

$$+ \frac{1}{2} g^{\mu\sigma} \left(g_{\mu\sigma,\lambda} + \cancel{g_{\lambda\sigma,\mu}} - \cancel{g_{\mu\lambda,\sigma}} \right) T^\lambda{}_\nu$$

$$- \frac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + \cancel{g_{\nu\sigma,\mu}} - \cancel{g_{\mu\nu,\sigma}} \right) T^\mu{}_\lambda$$

$$= \partial_\mu T^\mu{}_\nu + \frac{1}{2} g^{\mu\sigma} (g_{\mu\sigma,\lambda}) T^\lambda{}_\nu - \frac{1}{2} g_{\mu\sigma,\nu} T^{\mu\sigma}$$

compare to right - hand - side

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \, T^{\mu}{}_{\nu} \right) - \frac{1}{2} T^{\alpha\beta} \left(g_{\alpha\beta,\nu} \right)$$

$$= \partial_\mu T^{\mu}{}_{\nu} + \frac{1}{2} g^{\alpha\beta} \left(g_{\alpha\beta,\mu} \right) T^{\mu}{}_{\nu} - \frac{1}{2} T^{\alpha\beta} \left(g_{\alpha\beta,\nu} \right)$$

$$= \text{LHS}$$

note that we used a variant of eq. (3) replacing $\delta \to \partial_\mu$

$$\partial_\mu \sqrt{-g} \overset{3b}{=} \frac{1}{2} \sqrt{-g} \, g^{\alpha\beta} \left(\partial_\mu g_{\alpha\beta} \right)$$

c) Start from eq. (9) and substitute (12):

$$\delta S^{\text{matter}} = \frac{1}{2} \int d^4x \sqrt{-g} \, T^{\mu\nu} \, \delta g_{\mu\nu}$$

$$= \frac{1}{2} \int d^4x \sqrt{-g} \, T^{\mu\nu} \left(\xi^\alpha g_{\mu\nu,\alpha} + \left(\partial_\mu \xi^\alpha \right) g_{\alpha\nu} + \left(\partial_\nu \xi^\alpha \right) g_{\mu\alpha} \right)$$

$$= \frac{1}{2} \int d^4x \sqrt{-g} \, T^{\mu\nu} \left(\xi^\alpha g_{\mu\nu,\alpha} + 2 g_{\alpha\mu} \left(\partial_\nu \xi^\alpha \right) \right)$$

$$\overset{\text{P.I.}}{=} \int d^4x \sqrt{-g} \, \xi^\alpha \left\{ \frac{1}{2} T^{\mu\nu} g_{\mu\nu,\alpha} - \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} \, T^{\nu}{}_{\alpha} \right) \right\}$$

$$= - \int d^4x \sqrt{-g} \, \xi^\alpha \left\{ \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \, T^{\mu}{}_{\alpha} \right) - \frac{1}{2} T^{\mu\nu} g_{\mu\nu,\alpha} \right\}$$

$$= - \int d^4x \sqrt{-g} \, \xi^\alpha \left(D_\mu T^{\mu}{}_{\alpha} \right)$$

□

d) If δ_c corresponds to a coordinate transformation and S^{matter} is invariant, we have $\delta_c S^{matter} = 0$

Since ξ^α is an arbitrary function, this can only be true if

$$D_\mu T^\mu{}_\alpha = 0$$

$\Rightarrow$ conservation law for the energy momentum tensor based on symmetry principles.