Cosmology : problem set : week 3

Solutions :

Exercise 1 : (handin !)

a) The light ray moves in the orbital plane

$$\theta = \frac{\pi}{2} \,, \qquad \frac{d\theta}{d\lambda} = 0$$

Spelling out the line - element (3) gives

$$0 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2$$

$$+ \, r^2 \underbrace{\left(\frac{d\theta}{d\lambda}\right)^2}_{=0 \text{ by choice of coordinates}} + \, r^2 \sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2$$

Substituting the conserved quantities

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} \,, \qquad \ell = r^2 \underbrace{\sin^2\theta}_{= 1 \text{ for } \theta = \frac{\pi}{2}} \frac{d\phi}{d\lambda}$$

and the information on the orbit yields :

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\,\ell^2$$

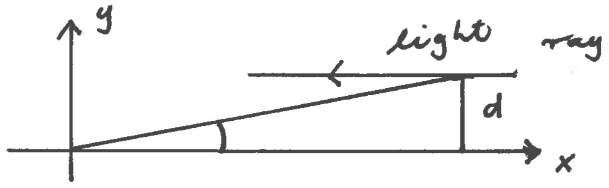
equation (4) is obtained by multiplying this relation

by $\left(1 - \frac{2M}{r}\right) \cdot \frac{1}{\ell^2}$:

gives

$$\frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{1}{r^2}\left(1-\frac{2M}{r}\right)}_{= \, W_{eff}\,.} \qquad (1)$$

b) Geometric analysis:



far away from the source ($r \to \infty$; $x \to \infty$) we have

$$\tan\phi = \frac{d}{x} \simeq \frac{d}{r} \simeq \phi$$

Thus infinitesemally

$$\frac{d\phi}{d\lambda} = -\frac{d}{r^2}\frac{dr}{d\lambda}$$

Evaluate the line element in the limit $r \to \infty$

$$0 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$= -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{d^2}{r^2}\left(\frac{dr}{d\lambda}\right)^2$$

lim $r \to \infty$

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2$$

Pick the new curve parameter t :

$$0 = -1 + \left(\frac{dr}{dt}\right)^2$$

$\Rightarrow$

$$\frac{dr}{dt} = \pm 1 \qquad (2)$$

fix sign:

r decreases as t increases (ingoing light ray):

to reflect this, the "-" sign in the relation above must be chosen:

$$\frac{dr}{dt} = -1 \qquad (3)$$

Then use the geometric identity and take $\lambda = \lambda(t)$

$$\frac{d\phi}{dt} = -\frac{d}{r^2} \underbrace{\frac{dr}{d\lambda}\frac{d\lambda}{dt}}_{= -1}$$

Thus

$$\frac{d\phi}{dt} = \frac{d}{r^2} \qquad (4)$$

Since e, ℓ are conserved quantities, they do not change along the world line.

We then have

$$b = \lim_{r \to \infty} \left| \frac{\ell}{e} \right| = \frac{r^2 \dfrac{d\phi}{d\lambda}}{\left(1 - \dfrac{2M}{r}\right) \dfrac{dt}{d\lambda}}$$

$$= \lim_{r \to \infty} \frac{r^2 \dfrac{d\phi}{d\lambda}}{\left(1 - \dfrac{2M}{r}\right) \dfrac{dt}{d\lambda}} = r^2 \frac{d\phi}{d\lambda} \frac{d\lambda}{dt}$$

$$\overset{(4)}{=} d$$

c) Use the chain rule to write:

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi} \frac{d\phi}{d\lambda} = \frac{dr}{d\phi} \cdot \frac{\ell}{r^2}$$

Substitute into eq. 1:

$$\frac{1}{b^2} = \frac{1}{\ell^2} \cdot \frac{\ell^2}{r^4} \left(\frac{dr}{d\phi} \right)^2 + W_{eff}$$

$$\left(\frac{dr}{d\phi} \right)^2 = r^4 \left(\frac{1}{b^2} - W_{eff} \right)$$

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2} \left(\frac{1}{b^2} - W_{eff} \right)^{-1/2}$$

note : at the turning point r_1, the light ray has been bent by the angle $\Delta\phi/2$. Since the problem is symmetric around r_1 (ingoing - outgoing light ray) the full deflection angle is found as:

$$\Delta \phi = 2 \int_{r_1}^{\infty} \frac{dr}{r^2} \left[1/b^2 - W_{eff}(r) \right]^{-1/2} \qquad (5)$$

where r_1 corresponds to $\frac{d\phi}{dr} = 0$

d) Change of integration variable:

$$w = \frac{b}{r} \quad , \quad dw = - \frac{b}{r^2} \, dr$$

$$W_{eff}(w) = \frac{1}{b^2} w^2 \left(1 - \frac{2M}{b} w \right)$$

$$r \to \infty \; , \quad w = 0 \quad ; \quad r_1 \to w_1$$

Substitute into (5):

$$\Delta \phi = 2 \int_{w_1}^{0} \left(- \frac{dw}{b} \right) \cdot b \left[1 - w^2 \left(1 - \frac{2Mw}{b} \right) \right]^{-1/2}$$

$$= 2 \int_0^{w_1} dw \left[1 - w^2 \left(1 - \frac{2Mw}{b} \right) \right]^{-1/2}$$

e) Expansion

We write the denominator:

$$\left(1 - w^2\left(1 - \frac{2M}{b} w\right)\right)^{-1/2}$$

$$= \left(1 - \frac{2M}{b} w\right)^{-1/2} \left(\left(1 - \frac{2M}{b} w\right)^{-1} - w^2\right)^{-1/2}$$

$$\simeq \left(1 + \frac{M}{b} w\right)\left(1 + \frac{2M}{b} w - w^2\right)^{-1/2}$$

Thus the approximate integral is

$$\Delta\phi \approx \int_0^{w_+} dw \, \frac{\left(1 + \frac{M}{b} w\right)}{\left(1 + \frac{2M}{b} w - w^2\right)^{1/2}}$$

f) The integral above is tabulated (see e.g. Teubner)

To determine the analytic branch of the solution

set

$$Q = a x^2 + \tilde{b} x + c$$

$$D = 4 a c - \tilde{b}^2$$

$$d = \frac{4a}{D}$$

The analytic branch of the solution is determined by the sign of the determinant:

$$D = -4 - \tilde{b}^2 < 0$$

In this case

$$\int \frac{dx}{\sqrt{Q}} = - \frac{1}{\sqrt{-a}} \arcsin \frac{2ax + \tilde{b}}{\sqrt{-D}} \qquad \begin{array}{l} a < 0 \\ D < 0 \end{array}$$

and

$$\int \frac{x\,dx}{\sqrt{Q}} = \frac{\sqrt{Q}}{a} - \frac{\tilde{b}}{2a} \int \frac{dx}{\sqrt{Q}}$$

Evaluate the integral:

$$\Delta\phi = 2 \left[\int \frac{dx}{\sqrt{Q}} + \frac{\tilde{b}}{2} \int \frac{x\,dx}{\sqrt{Q}} \right]$$

$$= 2 \left[\int \frac{dx}{\sqrt{Q}} + \frac{\tilde{b}}{2} \frac{\sqrt{Q}}{a} - \frac{\tilde{b}^2}{4a} \int \frac{dx}{\sqrt{Q}} \right]$$

substitute explicit expressions:

$$= 2 \left(- \frac{M}{b} \sqrt{Q} \Big|_0^{w_+} \right) + 2 \left(1 + \frac{M^2}{b^2} \right) \int \frac{dx}{\sqrt{Q}}$$

$$= \frac{2M}{b} + 2 \left(1 + \frac{M^2}{b^2} \right) (-1) \arcsin 2 \frac{M/b - x}{\sqrt{4 + \frac{4M^2}{b^2}}} \Bigg|_0^{w_+}$$

$$= \frac{2M}{b} - 2 \left(1 + \frac{M^2}{b^2} \right) \left[\underbrace{\arcsin(-1)}_{= -\frac{\pi}{2}} - \underbrace{\arcsin \frac{M/b}{\sqrt{1 + M^2/b^2}}}_{= \frac{M}{b} + \dots} \right]$$

$$= \pi + \frac{4M}{b} + O(M/b)^2$$

using the definition of the deflection angle:

$$\delta\phi = \Delta\phi - \pi$$

$$= \frac{4M}{b}$$

$\square$

Exercise 2:

a) The geodesic equation for the particle following the world line $x^\alpha(\lambda)$ is

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

For the t-component ($\alpha = 0$) there is only one non-zero Christoffel symbol contributing

$$\Gamma^t{}_{tr} = \left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)^{-1}$$

Taking into account the symmetry $\Gamma^t{}_{tr} = \Gamma^t{}_{rt}$ the geodesic equation yields

$$\frac{d^2t}{d\lambda^2} = -\left(\Gamma^t{}_{rt}\frac{dr}{d\lambda}\frac{dt}{d\lambda} + \Gamma^t{}_{tr}\frac{dt}{d\lambda}\frac{dr}{d\lambda}\right)$$

$$= -2\left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\lambda}\frac{dt}{d\lambda}$$

b) Take the λ-derivative of the conserved quantity e:

$$\frac{de}{d\lambda} = \left(1-\frac{2M}{r}\right)\frac{d^2t}{d\lambda^2} + \left(\frac{2M}{r^2}\right)\frac{dr}{d\lambda}\frac{dt}{d\lambda}$$

$$= \left(1-\frac{2M}{r}\right)\left(\frac{d^2t}{d\lambda^2} + \left(1-\frac{2M}{r}\right)^{-1}\frac{2M}{r^2}\frac{dr}{d\lambda}\frac{dt}{d\lambda}\right)$$

- the second bracket is precisely the geodesic equation
- thus e is constant along the world line.