Werkcollege, Cosmology 2016/2017, Week 6

These are the exercises and hand-in assignment for the 6th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 6.1** below.

6.1 Mass distribution of the Milky Way

a. Assuming that the mass distribution in the Milky Way is dominated by a spherically symmetric dark matter halo, show that a flat rotation curve implies the following density profile:

$$\rho_h(R) = \frac{v_c^2}{4\pi G} R^{-2} \tag{6.1.1}$$

where R is the galactocentric distance and v_c the circular velocity.

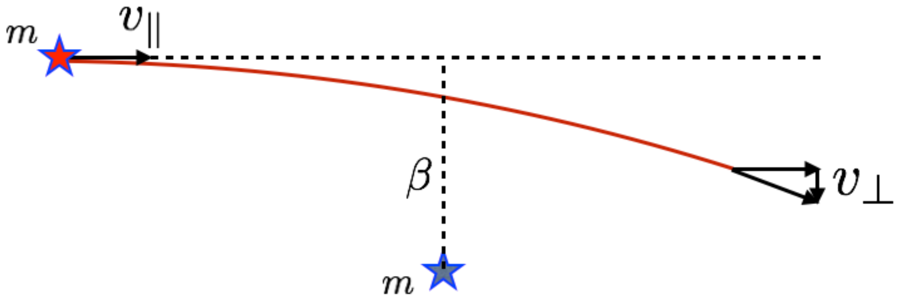
b. For a circular velocity $v_c = 200$ km/s, $R_0 = 8$ kpc, calculate the density of the dark matter halo near the Sun.

In reality, other components of the Milky Way make non-negligible contributions to the mass. Near the Sun, the density of the stellar disc is about $\rho_d(R_0) = 0.08\,M_\odot\,\mathrm{pc}^{-3}$. Assume that the Sun is located at the midplane of the disc and that the vertical density distribution of the disc is exponential with scale height $z_{\rm scl} = 300$ pc.

c. At R_0, how far above the Galactic plane, z, is the density of the disc equal to that of the dark halo estimated above? (you may assume that ρ_h is independent of z for fixed $R = R_0$). If you did not find an answer in **6.1.b** you may assume $\rho_h(R_0) = 0.02\,M_\odot\,\mathrm{pc}^{-3}$ but note that this is *not* the correct answer.

6.2 Two-body relaxation

The process of two-body relaxation plays a very important role in stellar dynamics. Over time, it drives the distribution of stellar velocities towards a Maxwellian equilibrium distribution, so that any memory of the initial conditions will eventually be erased. For the typical stellar densities and relative velocities encountered in galaxies the two-body relaxation time scale is, however, very long, so that present-day galaxies still retain some memory of their formation conditions. In this exercise we go through the derivation of the two-body relaxation time scale.



Consider an encounter between two stars. Assume for simplicity that both stars have the same mass, m. We use a coordinate system in which one star is initially moving along a straight line with velocity $v_{||}$, equal to the typical relative velocities V of stars in the system, and the other is stationary. Continuing along this path, the minimum separation between the two stars ((the *impact parameter*) will be β (see figure), and the star will experience an acceleration a due to the mutual gravitational attraction between the two stars. Clearly, by Newton's 3rd law, the other star will experience an acceleration of the same magnitude but opposite direction. The component of a perpendicular to $v_{||}$, $a_{\perp}$, will produce a net velocity $v_{\perp}$ perpendicular to $v_{||}$ after the encounter.

One distinguishes between *strong* and *weak* encounters, where an encounter is said to be *strong* if the smallest distance of the stars during the encounter (β) is such that the (absolute) potential energy $|U(\beta)|$ is equal to (or greater than) the mean kinetic energy of a star.

a. Show that a strong encounter corresponds to an impact parameter

$$\beta < \frac{2Gm}{V^2} \tag{6.2.1}$$

In the solar neighbourhood, the mean volume density of stars is about $n = 0.1\ \mathrm{pc}^{-3}$. Typical relative velocities are 10 km/s, and the average mass of a star can be taken to be $m = 1M_{\odot}$.

b. Show that the mean rate of strong encounters per star is

$$\frac{\mathrm{d}n_{\mathrm{enc}}}{\mathrm{d}t} = 4\pi G^2 n m^2 V^{-3} \tag{6.2.2}$$

Hence, demonstrate that the Sun is unlikely to have experienced a strong encounter in its lifetime.

From the above, it follows that most stellar encounters are of the *weak* type. This means that the velocity change of a star, during any one encounter, is typically small ($v_{\perp} \ll v_{||}$). It is the cumulative effect of many *distant* encounters that will, eventually, be important. For further calculations, we will thus evaluate the forces and accelerations as if the first star continues moving along the original path and the second star remains stationary.

c. Under these assumptions, show that the acceleration of the star perpendicular to $v_{\|}$, integrated over all positions along the path, produces a perpendicular velocity

$$v_{\perp} = 2\frac{Gm}{\beta v_{\|}} \tag{6.2.3}$$

You may find the following integral useful:

$$\int \frac{\mathrm{d}x}{X\sqrt{X}} = 2\frac{2ax+b}{\Delta\sqrt{X}} \tag{6.2.4}$$

where $X \equiv ax^2 + bx + c$ and $\Delta = 4ac - b^2$.

For relative velocities $V \sim v_{\|}$ and stellar density n, the number of encounters with impact parameter between β and $\beta + \mathrm{d}\beta$ in a small time step $\mathrm{d}t$ will be

$$\mathrm{d}^2 N_{\mathrm{enc}} = 2\pi\beta n V\,\mathrm{d}\beta\,\mathrm{d}t \tag{6.2.5}$$

Since the encounters may occur in random directions, the total effect of many encounters (ΔV) is found by adding the contributions of each encounter (6.2.3) quadratically,

$$\Delta V^2 = \sum v_{\perp}^2 \tag{6.2.6}$$

d. Hence show, by integrating over impact parameters in a range $\beta_{\mathrm{min}} < \beta < \beta_{\mathrm{max}}$, that the total (average) velocity change in a small time step $\mathrm{d}t$ is

$$\langle \mathrm{d}V^2 \rangle = \frac{8\pi G^2 m^2 n}{V} \ln\left(\frac{\beta_{\mathrm{max}}}{\beta_{\mathrm{min}}}\right) \mathrm{d}t \tag{6.2.7}$$

It is not obvious what to pick for β_{min} and β_{max}, but since only the logarithm of the ratio of these two quantities enters in the expression, their exact values are not important. Usually, it is reasonable to assume $\ln\Lambda \equiv \ln\left(\frac{\beta_{\mathrm{max}}}{\beta_{\mathrm{min}}}\right) \approx 10$. The quantity $\ln\Lambda$ is also known as the *Coulomb logarithm*.

Finally, the *two-body relaxation time scale*, t_{relax}, is now defined as the time that it takes for the effect of the cumulative distant encounters to produce a velocity change similar to the average relative velocities of the stars, $\langle \mathrm{d}V^2 \rangle = V^2$.

e. Assuming that the density and average relative velocities are constant in time, show that this is now given as

$$t_{\mathrm{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln\Lambda} \tag{6.2.8}$$

which is the expression discussed in the lecture.

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5 \log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

Mass of the Sun: $M_\odot = 2 \times 10^{30}$ kg

1 pc = 3.09×10^{16} m

Planck's constant: $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}

Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Gravitational constant: $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}