Werkcollege, Cosmology 2016/2017, Week 12

These are the exercises and hand-in assignment for the 12th week of the course *Cosmology*. Every week, one of the problems provides credit towards the final exam. If at least **10** of these problems are handed in and approved, one problem on the final exam may be skipped. The hand-in assignment for this week is **Problem 12.2** below.

12.1 Cosmological surface brightness dimming (Wed)

In astronomy, the *luminosity* L of a source is the energy output per unit time (e.g. measured in W), the *flux* is the energy passing through a surface of unit area per unit time (e.g. in units of W m^{-2}) and the *intensity* I of radiation is the flux per unit solid angle (W m^{-2} sr^{-1}). It is straight forward to show that the intensity is distance-independent in standard Euclidian geometry, as long as there is no absorbing material between the source and the observer.

- Using the definitions of angular diameter- and luminosity distance, show that the intensity of a source decreases with redshift as

$$I(z) = I_0(1+z)^{-4} \tag{12.1.1}$$

12.2 Cosmological distances (adapted from Reexam 2013/2014) (Wed)

Recall that a line element in the Friedman-Robertson-Walker metric may be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a^2(t)}{c^2}\left[\mathrm{d}r^2 + \mathcal{R}^2 \sin^2(r/\mathcal{R})(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{12.2.1}$$

for scale factor a, co-moving radial coordinate r, and radius of curvature $\mathcal{R}$.

We have seen that it is useful to define the *angular diameter distance*, D_A, as

$$D_A = \frac{D}{1+z} \tag{12.2.2}$$

for distance measure

$$D = \mathcal{R}\sin(r/\mathcal{R}). \tag{12.2.3}$$

With this definition, we then have following relation between the length dl of a standard rod, oriented perpendicular to the line-of-sight, the apparent angular size of the rod dθ, and D_A:

$$\mathrm{d}l = D_A \mathrm{d}\theta \tag{12.2.4}$$

which is similar to the usual Euclidian relation.

The general expression for the comoving radial coordinate, r, is

$$r = \int_{t_1}^{t_0} \frac{c}{a(t)}\mathrm{d}t \tag{12.2.5}$$

for light emitted from a source at $t = t_1$ and received by an observer at $t = t_0$. In general, this expression must be integrated numerically, although analytic solutions are possible in some cases. Here we explore one such case, the *Einstein-de Sitter Universe*.

In an *Einstein-de Sitter Universe*, $\Omega_0 = 1$ and $\Omega_\Lambda = 0$. For this particular case, the cosmic time t, the Hubble constant H_0, and the scale factor $a(t)$ are related as:

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3} \tag{12.2.6}$$

a. Show that, for an Einstein-de Sitter Universe, the comoving radial coordinate r and the redshift z are related as

$$r = \frac{2c}{H_0}\left(1 - (1+z)^{-1/2}\right) \tag{12.2.7}$$

Hint: The following integral may come in handy:

$$\int_0^a (1+x)^{-3/2}\,\mathrm{d}x = 2\left(1 - (1+a)^{-1/2}\right) \tag{12.2.8}$$

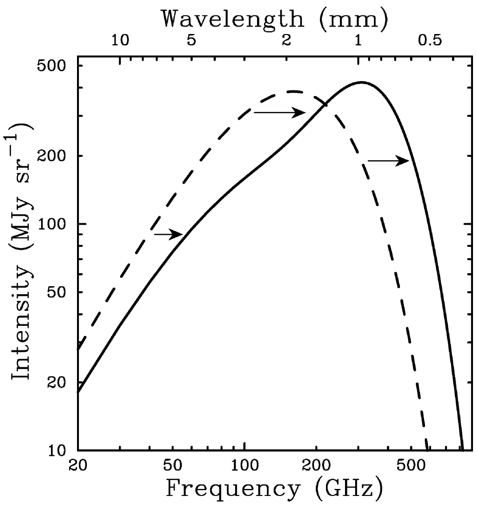
b. Show that D_A has an extremum at $z = 5/4$ in the Einstein-de Sitter Universe, and argue that this must be a maximum. What does this imply for the apparent sizes of objects (of a given linear size) as a function of redshift?

12.3 The Sunyaev-Zeldovich effect (Thu)

In the Sunyaev-Zeldovich effect, CMB photons are inverse Compton scattered to higher energies when passing through hot gas in galaxy clusters. The energy increment is

$$\Delta E_\nu / E_\nu = y \tag{12.3.1}$$

where y is the Compton optical depth. This is illustrated schematically in the figure below:



At low frequencies ($h\nu \ll kT$), the CMB black-body spectrum can be approximated by the Rayleigh-Jeans formula,

$$I_\nu \approx \frac{2\nu^2 kT}{c^2} \tag{12.3.2}$$

a. Convince yourself that an energy boost of the form (12.3.1) corresponds to a purely horizontal shift of the CMB spectrum (when plotted as I_ν, i.e. specific intensity per frequency interval).

b. Then show that in the Rayleigh-Jeans limit, the decrease in the observed intensity is

$$\Delta I_\nu / I_\nu = -2y \tag{12.3.3}$$

Formulae and constants

Distance modulus (D in pc):

$$m - M = 5 \log_{10} D - 5$$

Black-body radiation:

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

Radius of the Sun: $R_\odot = 7 \times 10^8$ m

Mass of the Sun: $M_\odot = 2 \times 10^{30}$ kg

1 pc = 3.09×10^{16} m

Planck's constant: $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}

Boltzmann's constant: $k = 1.38 \times 10^{-23}$ m^2 kg s^{-2} K^{-1}

Gravitational constant: $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}