ASSIGNMENTS Week 16 (F. Saueressig)

Cosmology 16/17 (NWI-NM026C)

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Exercise 4 is a hand-in assignment. Please submit your solution to your teaching assistant at the end of the tutorial session on **Thursday, 12th January**. Late hand-ins will not be accepted since the solution will be published at the end of the week.

Exercise 1: cosmology revisited

Give a concise answer to the following questions:

a) The contribution of relativistic particles to the total energy density of the universe is

$$\rho = g_* \frac{\pi^2}{30} T^4 \tag{1}$$

where T is the temperature of the cosmic plasma and g_* denotes the effective number of relativistic degrees of freedom.

1) Explain why bosonic and fermionic degrees of freedom contribute to g_* with a different weight.

2) Briefly explain why the value of g_* at dark matter creation (at $T \approx 200$GeV) differs from g_* at nucleosynthesis ($T \approx 1$ MeV).

b) When studying density fluctuations on a homogeneous and isotropic Friedmann-Robertson-Walker background at the linearized level, we extensively used that Fourier modes with different spatial wave number $\vec{k}$ do not interact. What is the reason for this decoupling?

c) The Bunch-Davies vacuum (first obtained by Chernikov and Tagirov) is the zero particle state seen by a geodesic observer in de Sitter space. Explain why this vacuum state plays an important role in inflationary cosmology.

d) When studying scalar field inflation, we introduced the potential slow-roll parameters

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2 , \quad \eta_V \equiv M_{\rm Pl}^2 \, \frac{V_{,\phi\phi}}{V} \, . \tag{2}$$

Here $M_{\rm Pl} \equiv (8\pi G)^{-1/2}$, $V(\phi)$ is the scalar potential and the comma denotes a derivative with respect to the scalar field. Slow-roll inflation is typically associated with the slow-roll parameters being small: $\epsilon_V \ll 1$, $\eta_V \ll 1$. Explain why these conditions are necessary but not sufficient to indicate that the system undergoes inflation.

Exercise 2: Slow-roll inflation revisited

The dynamics of a minimally coupled scalar field evolving in a Friedmann-Robertson-Walker (FRW) universe is governed by the equations of motion

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,, \qquad H^2 = \frac{M_{\rm Pl}^{-2}}{3}\left(\frac{1}{2}\,\dot{\phi}^2 + V(\phi)\right)\,. \tag{3}$$

Here $M_{\rm Pl} \equiv (8\pi G)^{-1/2}$ denotes the reduced Planck mass and the comma is short-hand for a derivative of the potential with respect to ϕ. In order to analyze the dynamics of inflation, one typically studies the potential slow-roll parameters ϵ_V and η_V. Analyze the single-field inflationary model, where the potential is given by a ϕ^4 term,

$$V(\phi) = \lambda\,\phi^4\,, \tag{4}$$

and λ is a positive coupling constant.

a) Determine the slow-roll parameters ϵ_V and η_V for this model.

b) Requiring the slow-roll conditions $\epsilon_{\rm V}, |\eta_{\rm V}| < 1$, determine the values of ϕ where inflation is supported. Taking the (standard) condition that inflation ends if $\epsilon_{\rm V} = 1$ or $|\eta_{\rm V}| = 1$, determine $\phi_{\rm end}$ in terms of $M_{\rm Pl}$.

c) Assuming that the scalar field starts with an initial value $\phi_{\rm init} > \phi_{\rm end}$ determine the number of e-folds of expansion the universe undergoes in this inflationary period.

d) Evaluating the formula

$$\frac{1}{M_{\rm Pl}}\int_{\phi_{\rm end}}^{\phi_{\rm cmb}} \frac{d\phi}{\sqrt{2\epsilon_{\rm V}}} = N_{\rm cmb} \tag{5}$$

for $N_{\rm cmb} \approx 60$, determine the value $\phi_{\rm cmb}$ at which the fluctuations in the cosmic microwave background (CMB) are created.

e) Compute the slow-roll parameters $\epsilon_{\rm V}$ and $\eta_{\rm V}$ at $\phi_{\rm cmb}$ and express the result in terms of $N_{\rm cmb}$.

f) The massless scalar power spectrum is normalized so that $\Delta_s^2 \sim 10^{-9}$. Determine the resulting value of λ.

g) Use the slow-roll approximation to compute the scalar spectral index n_s and the scalar-to-tensor ratio r predicted by this model.

Exercise 3: Non-equilibrium phases in the early universe (literature survey)
Consider the following "events" in the cosmic history:

1) the formation of hadrons (protons, neutrons, and their anti-particles)

2) the formation of leptons and anti-leptons

3) the creation of light elements (big bang nucleosynthesis)

4) the creation of the cosmic microwave background (CMB)

What is the reaction that falls out of equilibrium at these events? Attribute a typical time- and energy scale to these transitions.

Exercise 4: Tracking the neutron fraction X_n numerically (hand-in exercise)
The neutron fraction is defined as

$$X_n = \frac{n_n}{n_n + n_p} \, . \tag{6}$$

Defining the evolution variable $x \equiv \mathcal{Q}/T$, where $\mathcal{Q} = m_n - m_p = 1.293$ MeV and T is the temperature of the cosmic plasma, the Boltzmann equation provides the following first-order differential equation determining X_n as a function of x:

$$\frac{dX_n}{dx} = \frac{x \, \lambda_{np}}{H(x=1)} \left[e^{-x} - X_n \left(1 + e^{-x} \right) \right] \, . \tag{7}$$

Here $H = 1.13 s^{-1}$ and the neutron-proton conversion rate is given by

$$\lambda_{np} = \frac{255}{\tau_n \, x^5} (12 + 6x + x^2) \tag{8}$$

with $\tau_n = 886.7$s being the neutron decay time.

a) Determine initial conditions for (7) at temperature $T = 5\mathcal{Q}$. What is the corresponding value of $x = x_{\text{init}}$ and the neutron fraction X_n? Hint: The neutron fraction may be computed from the number densities of a species in thermal equilibrium. You can set the chemical potential $\mu = 0$ and approximate $m_n = m_p$ in the prefactor.

b) Use the initial conditions from a) and solve (7) numerically. Plot your result to check that $X_n(x)$ indeed freezes out for $x > 1$.

c) Use the numerical solution to determine the neutron fraction at temperature $T = 0.1$ MeV.