# ASSIGNMENTS Week 10 (F. Saueressig)

# Cosmology 16/17 (NWI-NM026C)

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**Exercise 1 is a hand-in assignment**. Please submit your solution to your teaching assistant before the tutorial on **Wendsday, 23rd November**. Present your solution in a readable way.

### Exercise 1: Particles at thermal equilibrium (hand-in exercise)

According to statistical physics the energy distribution of particles in thermal equilibrium is given by the Fermi-Dirac (fermions) and the Bose-Einstein (bosons) distribution

$$f(E,T) = \frac{1}{e^{(E-\mu)/(k_B T)} \pm 1} \,. \tag{1}$$

Here $+$ is for fermions, $-$ for bosons, $T$ is the temperature, and $E = \sqrt{\vec{p}^2 + m^2}$ denotes the energy of the particle with mass $m$. Moreover, $\mu$ is a chemical potential, $g$ is a degeneracy factor (e.g., photons have two polarizations which are accounted for by setting $g = 2$) and $k_B$ denotes the Boltzmann constant. We will work with $k_B = 1$ in the following. Based on the distributions (1) one defines the

$$\begin{array}{ll} \text{number density} & n = g \int \frac{d^3p}{(2\pi)^3} \, f(\vec{p}) \\ \text{energy density} & \rho = g \int \frac{d^3p}{(2\pi)^3} \, E(\vec{p}) \, f(\vec{p}) \\ \text{pressure} & p = g \int \frac{d^3p}{(2\pi)^3} \, \frac{|\vec{p}|^2}{3E(\vec{p})} \, f(\vec{p}) \,. \end{array} \tag{2}$$

a) Consider the limit where $|\mu| \ll T$ and $m \ll T$. In this limit the integrals (2) may be approximated by setting $\mu = 0$ and $m = 0$. Show that, in this limit

$$n = \begin{cases} \frac{3}{4\pi^2}\zeta(3) g T^3 \,, & \text{fermions} \\ \frac{1}{\pi^2}\zeta(3) g T^3 \,, & \text{bosons} \end{cases} \tag{3}$$

$$\rho = \begin{cases} \frac{7}{8}\frac{\pi^2}{30} \, g T^4 \,, & \text{fermions} \\ \frac{\pi^2}{30} \, g T^4 \,, & \text{bosons} \end{cases} \tag{4}$$

$$p = \frac{1}{3}\rho \approx \begin{cases} 1.0505 \, n T \,, & \text{fermions} \\ 0.9004 \, n T \,, & \text{bosons} \end{cases} \tag{5}$$

Here $\zeta(z) \equiv \sum_{n=1}^{\infty} n^{-z}$ is the Riemann zeta function and $\zeta(3) \approx 1.202$.

b) Consider the non-relativistic limit, $T \ll m$ and $T \ll m - \mu$. In this limit the typical kinetic energies are much below the mass $m$, so that one can approximate $E \approx m + \vec{p}^2/(2m)$. The second condition, $T \ll m - \mu$ leads to occupation numbers $\ll 1$ so that one considers a dilute system. In this case, the denominator appearing in (1) can be approximated by

$$e^{(E-\mu)/T} \pm 1 \approx e^{(E-\mu)/T} \tag{6}$$

and the expressions for bosons and fermions become approximately equal. Show that in this case

$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}} \tag{7}$$

$$\rho = n \left(m + \frac{3T}{2}\right) \tag{8}$$

$$p = n\,T \ll \rho\,. \tag{9}$$

c) Use the Friedmann equation for a spatially flat universe ($k = 0$) to obtain a relation between the Hubble parameter $H$ and the temperature of the universe. Show that in the radiation dominated era

$$H^2 = \frac{8\pi G}{3} \frac{\pi^2}{30} \left(g_b + \frac{7}{8} g_f\right) T^4\,, \tag{10}$$

where $g_b$ and $g_f$ count the bosonic and fermionic degrees of freedom which are still relativistic at temperature $T$ (and not frozen out).

d) Denote the number of effective relativistic degrees of freedom by $g_*$, i.e., $g_* \equiv \left(g_b + \frac{7}{8} g_f\right)$. Note that $g_*(T)$ depends on the temperature. Estimate $g_*(T)$ at the energy scale relevant for nucleosynthesis where $T \approx 1$ MeV.

**Exercise 2: Equation of state parameter for photons**

The general form of the stress-energy tensor of a perfect fluid is of the form

$$T^{\mu\nu} = (\rho + p)\, u^\mu\, u^\nu + p\, g^{\mu\nu}\,, \tag{11}$$

where $u^\mu$ is the four-velocity of the perfect fluid.

a) Defining the trace of the stress-energy tensor $T \equiv g^{\mu\nu} T_{\mu\nu}$ show that $T = -\rho + 3p$.

b) The action of a free photon minimally coupled to gravity has the form

$$S^\gamma = -\frac{1}{4} \int d^4x \sqrt{-g}\, F_{\mu\nu}\, F^{\mu\nu}\,, \tag{12}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor known from electrodynamics and indices are raised with the spacetime metric. Apply the variation formula with respect to the metric to construct the stress-energy tensor of photons in terms of the field strength $F_{\mu\nu}$ by explicitly constructing

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S^\gamma}{\delta g^{\mu\nu}}\,. \tag{13}$$

Hint: you may want to compare your result to the literature (see, e.g. Weinberg's book on cosmology).

c) Trace the stress-energy tensor constructed in part b) and show that $T = 0$. Combine your result with part a) to determine the equation of state-parameter for photons (more general, for relativistic particles).

**Exercise 3: Slow-roll parameters for single-field inflation**

The dynamics of a minimally coupled scalar field evolving in a Friedmann-Robertson-Walker (FRW) universe is governed by the equations of motion

$$\begin{aligned} &\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,, \\ &H^2 = \frac{M_{\rm Pl}^{-2}}{3}\left(\frac{1}{2}\,\dot{\phi}^2 + V(\phi)\right)\,. \end{aligned} \tag{14}$$

Here $M_{\rm Pl} \equiv (8\pi G)^{-1/2}$ denotes the reduced Planck mass and the comma is short-hand for a derivative of the potential with respect to $\phi$. In order to test if a model undergoes an inflationary phase one introduces the slow-roll parameters

$$\varepsilon \equiv -\frac{\dot{H}}{H^2}\,, \qquad \eta \equiv \epsilon - \frac{1}{2\epsilon}\frac{d\epsilon}{dN} = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{15}$$

where $\varepsilon < 1$ ensures that the universe undergoes accelerated expansion and $|\eta| < 1$ guarantees that the fractional change of $\varepsilon$ per $e$-fold of expansion $N \equiv \ln \frac{a_{\rm end}}{a}$ is small.

During a phase of slow-roll inflation, the potential energy of the scalar field dominates over the kinetic energy,

$$\dot{\phi}^2 \ll V(\phi)\,, \tag{16}$$

and the sustained accelerated expansion is ensured by the condition that the second time derivative of $\phi$ is sufficiently small

$$|\ddot{\phi}| \ll |3H\dot{\phi}|\,, \left|\frac{dV}{d\phi}\right|\,. \tag{17}$$

Show that the assumptions (16) and (17) allow to express the geometrical slow-roll parameters (15) in terms of the scalar potential

$$\varepsilon \simeq \epsilon_{\rm V}\,, \qquad \eta \simeq \eta_{\rm V} - \epsilon_{\rm V}\,. \tag{18}$$

where

$$\epsilon_{\rm V} \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2\,, \qquad \eta_{\rm V} \equiv M_{\rm Pl}^2\,\frac{V_{,\phi\phi}}{V}\,. \tag{19}$$

**Exercise 4: Single-field inflation: the case $m^2\phi^2$**

Carry out the slow-roll analysis for the, arguably, simplest single-field inflationary model where the potential is given by a mass term:

$$V(\phi) = \tfrac{1}{2}\,m^2\,\phi^2\,. \tag{20}$$

a) Determine the slow-roll parameters (19) for this model.

b) Requiring the slow-roll conditions $\epsilon_\mathrm{V}, |\eta_\mathrm{V}| < 1$, determine the values of $\phi$ where inflation is supported. Taking the (standard) condition that inflation ends if $\epsilon_\mathrm{V} = 1$ or $|\eta_\mathrm{V}| = 1$, determine $\phi_\mathrm{end}$.

c) Assuming that the scalar field starts with an initial value $\phi_\mathrm{init} > \phi_\mathrm{end}$ determine the number of $e$-folds of expansion the universe undergoes in this inflationary period.

d) Evaluating the formula

$$\frac{1}{M_\mathrm{Pl}} \int_{\phi_\mathrm{end}}^{\phi_\mathrm{cmb}} \frac{d\phi}{\sqrt{2\epsilon_\mathrm{V}}} = N_\mathrm{cmb} \tag{21}$$

for $N_\mathrm{cmb} \approx 40 - 60$, determine the scale at which the fluctuations in the cosmic microwave background (CMB) are created.

**Exercise 5: Reheating**

After inflation ends the scalar field begins to oscillate around the minimum of the potential. Close to the minimum the potential may then be approximated by a quadratic term, $V(\phi) = \frac{1}{2}m^2\phi^2$. Assuming that the oscillations are much faster than the evolution of the Hubble parameter, eq. (14) simplify to the equations of motion of an harmonic oscillator. Introducing the average energy $\bar{\rho}_\phi \equiv \langle \dot{\phi}^2 \rangle_t$ averaged over one oscillatory period to show that the averaged energy density decays as

$$\frac{d\bar{\rho}_\phi}{dt} + 3H\bar{\rho}_\phi = 0\,. \tag{22}$$