The Boltzmann equation for photons

The left-hand side of the Boltzmann equation can now be written in terms of **partial derivatives** for the *seven* independent variables (t, x^i, p^i):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}$$

magnitude of momentum   direction of momentum

The equilibrium distr. function (Bose-Einstein) depends only on *magnitude* of p, not the direction (isotropy).

Therefore $\partial f/\partial p^i$ is non-zero only for perturbed f, i.e. first order.
Similarly, $\mathrm{d}p^i/\mathrm{d}t$ is zero for non-perturbed metric. So the product,

$$\frac{\partial f}{\partial \hat{p}^i} \cdot \frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}$$

is a second-order term and can be neglected.

The Boltzmann equation for photons

We are left with: $$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Now consider the second term: $$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}$$

From definition of four-momentum,

$$\frac{\mathrm{d}x^0}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda} = P^0 \quad \text{and} \quad \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = P^i$$

We already found that $$P^0 = p(1-\Psi)$$

Similarly, for i=1..3, one finds $$P^i \approx p\hat{p}^i\frac{1-\Phi}{a}$$

So $$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{P^i}{P^0} = \frac{\hat{p}^i}{a}(1-\Phi)/(1-\Psi) \approx \frac{\hat{p}^i}{a}(1+\Psi-\Phi)$$

The Boltzmann equation for photons

We are left with: $$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} \approx \frac{\hat{p}^i}{a}(1 + \Psi - \Phi)$$

Again, $\partial f/\partial x^i$ is zero (f independent of x^i) for the unperturbed solution, so the terms involving $(\partial f/\partial x^i)(\Phi)$ and $(\partial f/\partial x^i)(\Psi)$ are second-order and can be neglected.

Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Next: $\mathrm{d}p/\mathrm{d}t$ …

The Boltzmann equation for photons

We now need the geodesic equation,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}$$

Using the definition of *P*,

$$P^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta$$

After some manipulation (exercise), one finds

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p \left\{ \frac{\partial \Psi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right\} - \Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{p} (1 + 2\Psi)$$

Evaluate Christoffel symbol, cancel more 2nd order terms (exercise)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -p \left(H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right)$$

The Boltzmann equation for photons

Inserting in the Boltzmann eqn, the left-hand side looks as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]$$

But we need a more concrete form of the distribution function *f(t,x,p)* to get further

Back to the distribution function

Unperturbed Bose-Einstein distribution:

$$f_{\rm BE} = \frac{1}{e^{E/T} - 1}$$

Introduce *perturbations* of temperature,

$$\Theta(x, \hat{p}, t) = \delta T(x, \hat{p}, t)/T(t)$$

so that

$$f(x, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1 + \Theta(x, \hat{p}, t)]}\right\} - 1\right]^{-1}$$

E=*p* for photons (*c*=1)

Expanding to first order in Θ gives (δT = TΘ)

$$f \approx f^{(0)} + T\frac{\partial f^{(0)}}{\partial T}\Theta = f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

The Boltzmann equation for photons

Left-hand side of Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]$$

Perturbed distribution function:

$$f \approx f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

The *zero-order* terms (no dep. on Θ, ϕ, ψ, x) yield

Equilibrium - collision terms in B.E. vanish

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_0 = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0$$

The Boltzmann equation for photons

The *zero-order* terms (no dep. on Θ, ϕ, ψ, x) yield

Equilibrium - collision terms in B.E. vanish

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_0 = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0$$

Using that $T\partial f^{(0)}/\partial T = -p\partial f^{(0)}\partial p$ (again), we have

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\partial f^{(0)}}{\partial p}\frac{p}{T}\frac{\mathrm{d}T}{\mathrm{d}t}$$

so

$$\left[-\frac{\mathrm{d}T/\mathrm{d}t}{T} - \frac{\mathrm{d}a/\mathrm{d}t}{a}\right]\frac{\partial f^{(0)}}{\partial p} = 0$$

$$\longrightarrow \quad T \propto a^{-1}$$

The Boltzmann equation for photons

The *first-order* terms are (exercise):

Collision terms due to Compton scattering

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] = C[f]$$

Now, the collision terms on the right-hand side.

Relevant physical process: Compton scattering,

$$e^-(q) + \gamma(p) \leftrightarrow e^-(q') + \gamma(p')$$

for electron momenta *q, q'* and photon momenta *p, p'*.

Collision terms for photons

Compton scattering:

$$e^-(q) + \gamma(p) \leftrightarrow e^-(q') + \gamma(p')$$

Schematically, the collision terms can be written as

$$\frac{\mathrm{d}f(\vec{p})}{\mathrm{d}t} = C[f(\vec{p})] = \sum_{\vec{q},\vec{q}',\vec{p}'} |\mathrm{Amplitude}|^2 \{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

"production" of photons with momentum p

"destruction" of photons with momentum p

Sum is written over all *q*, *q*', *p*', but energy and momentum must be conserv

Collision terms for photons

More formally, we have

Exercise 5 — Amplitude

$$C[f(\vec{p})] = \frac{1}{p}\int \frac{d^3\vec{q}}{(2\pi)^3 2E_e(q)} \int \frac{d^3\vec{q'}}{(2\pi)^3 2E_e(q')} \int \frac{d^3\vec{p'}}{(2\pi)^3 2E(p')} |\mathcal{M}|^2 (2\pi)^4$$

Integrals over all **q**, **q**', **p**'

$$\times\, \delta^3(\vec{p}+\vec{q}-\vec{p'}-\vec{q'}) \times \delta[E(p)+E_e(q)-E(p')-E_e(q')]$$

Momentum/energy conservation

$$\times\, [f_e(\vec{q'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})]$$

"Rate equation" (photons entering - leaving this bin of *f(p)*)

Notes:

Factors of *1/E* come from integration over *E* - taking into account that *E* and momenta are related through $E^2 = p^2 + m^2$, so that

$$\int d^3\vec{p} \int dE\, \delta(E^2 - p^2 - m^2) = \int \frac{d^3\vec{p}}{2E(p)}$$

(see also F. Saueressig's lectures)

Amplitude $|\mathcal{M}|^2$ depends on physics of Compton scattering

Collision terms for photons

First, concentrate on the integrals on the first line:

$$\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3 2E_e(q)} \int \frac{\mathrm{d}^3\vec{q'}}{(2\pi)^3 2E_e(q')} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2E(p')}$$

The electrons are non-relativistic at the epochs of interest: T ~ 3000 K at epoch of recombination, so $E_{\rm kin} = (3/2)kT \approx 0.26\,\mathrm{eV} \ll m_e c^2 (\approx 0.5\,\mathrm{MeV})$

We can therefore replace $E_e(q)$ with m_e in the denominators (setting c=1)

$$\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3 2E_e(q)} \approx \frac{1}{2m_e} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}$$

For photons, we have $E = p$, so

$$\int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2E(p')} = \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2p'}$$

Collision terms for photons

First, concentrate on the integrals on the first line:

$$\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3 2E_e(q)} \int \frac{\mathrm{d}^3\vec{q'}}{(2\pi)^3 2E_e(q')} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2E(p')}$$

$$\int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3 2E_e(q)} \approx \frac{1}{2m_e} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}$$

$$\int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2E(p')} = \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 2p'}$$

Inserting the integrals from the box, we then get:

$$C[f(\vec{p})] = \frac{(2\pi)^4}{8m_e^2 p} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{q'}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \delta^3(\vec{p} + \vec{q} - \vec{p'} - \vec{q'}) \times \delta[E(p) + E_e(q) - E(p') - E_e(q')]$$

$$\times [f_e(\vec{q'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p})]$$

Collision terms for photons

$$C[f(\vec{p})] = \frac{(2\pi)^4}{8m_e^2 p} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{q'}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \delta^3(\vec{p} + \vec{q} - \vec{p'} - \vec{q'}) \times \delta[E(p) + E_e(q) - E(p') - E_e(q')]$$

$$\times [f_e(\vec{q'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p})]$$

The middle integral is straight forward, making use of the momentum conservation:

$$\vec{q'} = \vec{p} + \vec{q} - \vec{p'}$$

Using the momentum delta function to evaluate the q' integral, we get

$$C[f(\vec{p})] = \frac{(2\pi)^4}{8m_e^2 p (2\pi)^3} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \delta[E(p) + E_e(q) - E(p') - E_e(\vec{p} + \vec{q} - \vec{p'})] \times [f_e(\vec{p} + \vec{q} - \vec{p'}) f(p') - f_e(q) f(p)]$$

- note that q' has now been eliminated.

Collision terms for photons

$$C[f(\vec{p})] = \frac{(2\pi)^4}{8m_e^2 p(2\pi)^3} \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \delta[E(p) + E_e(q) - E(p') - E_e(\vec{p}+\vec{q}-\vec{p'})] \times [f_e(\vec{p}+\vec{q}-\vec{p'})f(p') - f_e(q)f(p)]$$

- note that q' has now been eliminated!

Next, energy conservation: Photons: *E=p*, and (non-relativistic) electrons: $E = m_e + \frac{q^2}{2m_e}$

Also, since *q~q'* (change in electron momentum is small), we can replace *f_e(p+q-p')* with *f_e(q)* in the last factor. This leads to

$$C[f(p)] = \frac{\pi}{4m_e^2 p} \int \mathrm{d}^3\vec{q}\frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \left[\delta(p-p') + \frac{\partial\delta(p-p')}{\partial p'} \frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}\right] \times [f(\vec{p'}) - f(\vec{p})]$$

Collision terms for photons

$$C[f(p)] = \frac{\pi}{4m_e^2 p} \int \mathrm{d}^3\vec{q} \frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} |\mathcal{M}|^2$$

$$\times \left[\delta(p - p') + \frac{\partial \delta(p - p')}{\partial p'} \frac{\vec{q} \cdot (\vec{p} - \vec{p'})}{m_e} \right] \times [f(\vec{p'}) - f(\vec{p})]$$

Now, we need the amplitude term, $|\mathcal{M}|^2$:

For simplicity, we simply assume it to be constant: $|\mathcal{M}|^2 \approx 8\pi\sigma_T m_e^2$

(not strictly correct; depends on the angle between p and p', and on polarization, but final error is small)

Then we have

$$C[f(\vec{p})] = \frac{2\pi^2\sigma_T}{p} \int \mathrm{d}^3\vec{q} \frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} \times \left[\delta(p - p') + \frac{\partial \delta}{\partial p'} \frac{\vec{q} \cdot (\vec{p} - \vec{p'})}{m_e} \right] \times [f(\vec{p'}) - f(\vec{p})]$$

Collision terms for photons

$$C[f(\vec{p})] = \frac{2\pi^2\sigma_T}{p}\int \mathrm{d}^3\vec{q}\,\frac{f_e(\vec{q})}{(2\pi)^3}\int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} \times \left[\delta(p-p') + \frac{\partial\delta}{\partial p'}\frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}\right] \times [f(\vec{p'}) - f(\vec{p})]$$

Only two terms depend on *q*:

The integral of $f_e(q)$ over all *q* is the total electron density,

$$\int \frac{f_e(\vec{q})}{(2\pi)^3}\mathrm{d}^3\vec{q} = n_e$$

The integral of $q/m_e = v$ is the mean (bulk) electron velocity,

$$\int \frac{f_e(\vec{q})}{(2\pi)^3}\frac{\vec{q}}{m_e}\mathrm{d}^3\vec{q} = n_e v_b$$

And only the integral over p' is left:

$$C[f(p)] = \frac{2\pi^2 n_e\sigma_T}{p}\int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} \times \left[\delta(p-p') + \frac{\partial\delta}{\partial p'}\vec{v_b}\cdot(\vec{p}-\vec{p'})\right] \times [f(\vec{p'}) - f(\vec{p})]$$

Collision terms for photons

Integral over p':

$$C[f(p)] = \frac{2\pi^2 n_e \sigma_T}{p} \int \frac{\mathrm{d}^3\vec{p'}}{(2\pi)^3 p'} \times \left[\delta(p-p') + \frac{\partial\delta}{\partial p'}\vec{v_b}\cdot(\vec{p}-\vec{p'})\right] \times [f(\vec{p'}) - f(\vec{p})]$$

Integrating by parts takes care of the δ function (exercise).
Re-introduce our expansion of *f(p)*.

Define the *monopole* of the temperature perturbations as

$$\Theta_0(\vec{x},t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega' \Theta(\hat{p}',\vec{x},t)$$

This finally gives the result:

$$C[f(p)] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v_b}\right]$$

Putting it together:

Left-hand side of Boltzmann eq. (time derivative of distribution function)

$$\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_1 = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] = C[f]$$

Right-hand side (collision terms):

$$C[f(p)] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v_b}\right]$$

Combining the two, we finally have

$$\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b\right]$$

The Boltzmann equation for photons

$$\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b\right]$$

Replacing *t* with the *conformal* time ($c\eta$ = comoving horizon),

$$\eta \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')} \qquad\qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = 1/a(t)$$

so that $\qquad \dfrac{\partial\Theta}{\partial t} = \dfrac{\partial\Theta}{\partial\eta}\dfrac{\partial\eta}{\partial t} \equiv \dfrac{\dot{\Theta}}{a} \qquad$ etc..

the Boltzmann equation for photons finally becomes

$$\dot{\Theta} + \hat{p}^i\frac{\partial\Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T a\left[\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b\right]$$

The Boltzmann equation for photons

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a \left[\Theta_0 - \Theta + \hat{p} \cdot \mathbf{v}_b\right]$$

Partial differential equation, coupling variations in *temperature* distribution (Θ) to variations in the *potential* (ψ), *curvature* (ϕ) and *velocity* field ($\mathbf{v}_b$).

Simpler to solve in Fourier space, since
(a) partial derivatives are replaced by multiplication: $\mathcal{F}\left[f'(x)\right](k) = ik\mathcal{F}\left[f(x)\right](k)$

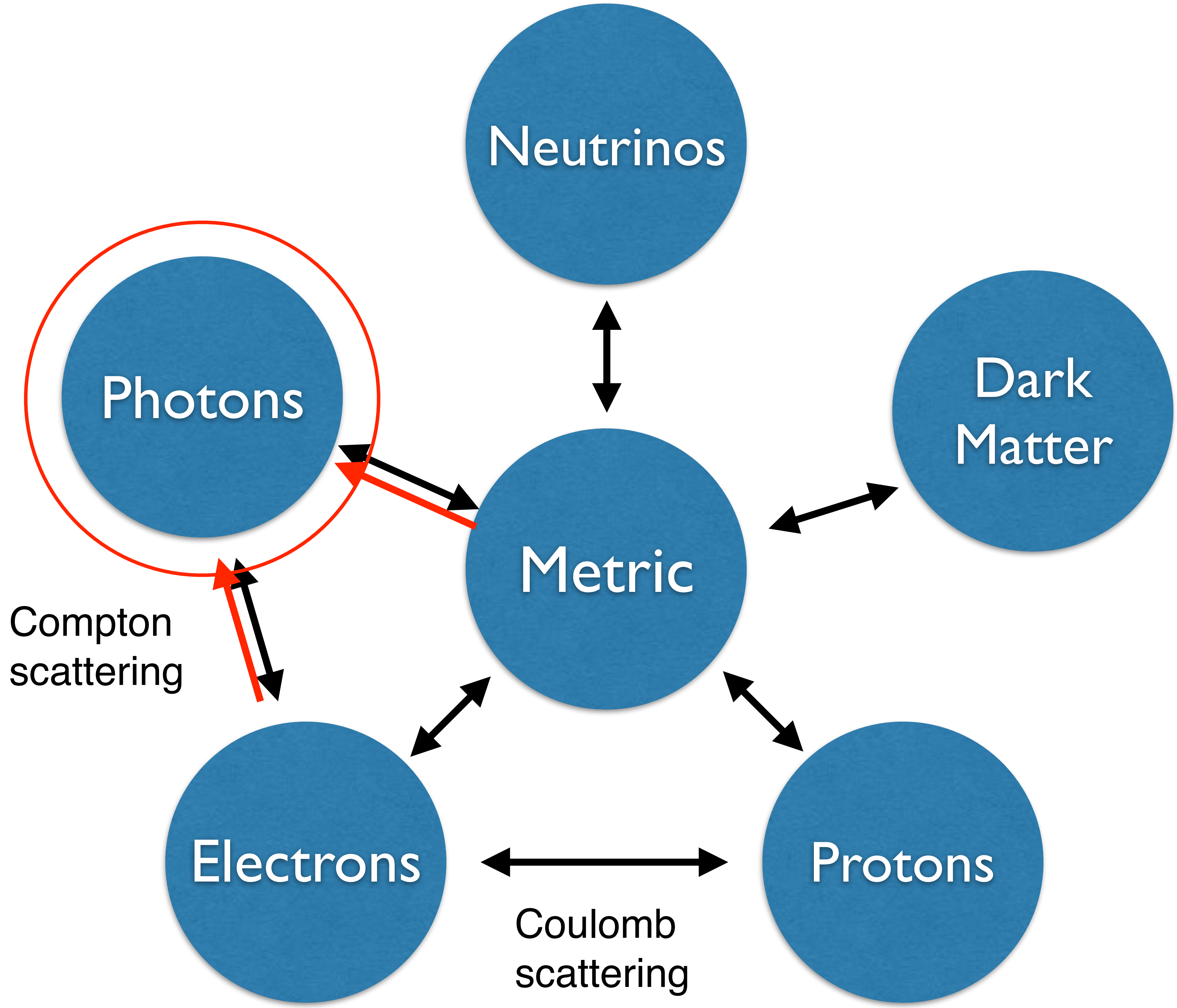
(b) small amplitude Fourier modes evolve *independently*.

Then we get

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = n_e \sigma_T a \left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu \mathbf{v}_b\right]$$

where $\mu \equiv \frac{\mathbf{k} \cdot \hat{p}}{k}$ is the direction of photon propagation w.r.t. the Fourier comp.

Ingredients and their coupling



Based on Dodelson, *Modern Cosmology*

B.E. for other components

We have derived the Boltzmann equation for *photons*.

Equivalent equations for

- **(Cold) dark matter (§4.5)**: no collision terms; particles are non-relativistic. No specific form for distrib. function assumed, use *moments* of B.E:

  Density fluctuations: $$\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0$$

  Velocity field: $$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0$$

- **Baryons (§4.6)**: Collision terms from Compton scattering;

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0$$

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = n_e\sigma_T a\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right]$$

- **Neutrinos**: similar to photons, but no collision terms

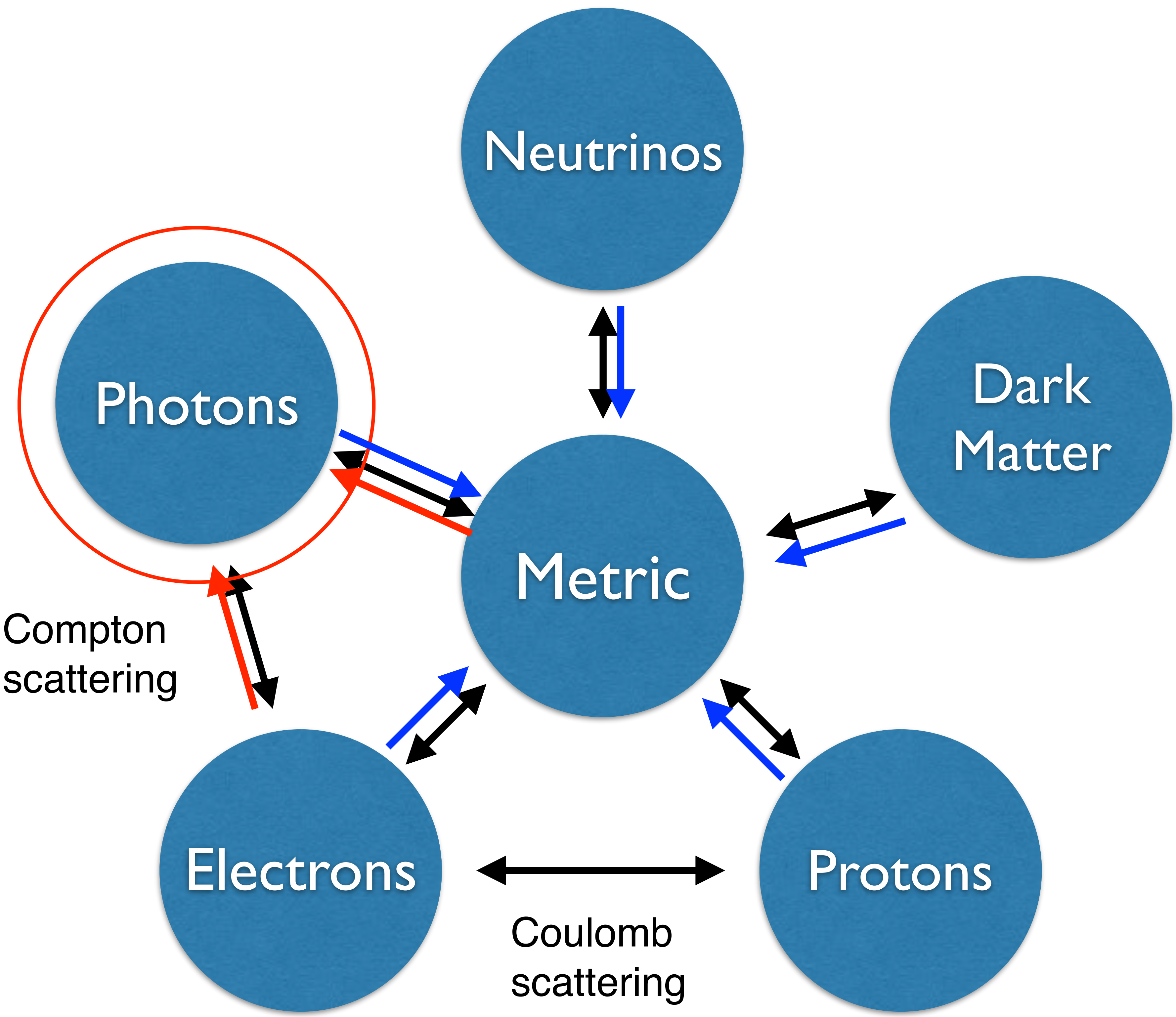
What we have got so far:

- A set of differential equations relating changes in the density, velocity, and temperature perturbations to the potential

Still missing:

- Finding out how the potential (i.e. the *metric*) responds to the perturbations

Ingredients and their coupling



Based on Dodelson, *Modern Cosmology*

The perturbed field equations

Einstein's field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu}$$

10 independent equations, but we need only two (to find ϕ and ψ).

Choose component (0,0)

$$G_{00} = 8\pi G T_{00}$$

It turns out to be useful (when evaluating T) to raise one of the indices:

$$\begin{aligned} G^0{}_0 &= g^{0i} G_{i0} \\ &= g^{00} G_{00} \\ &= g^{00} \left(R_{00} - \frac{1}{2} g_{00} \mathcal{R} \right) \\ &= (-1 + 2\Psi) R_{00} - \frac{\mathcal{R}}{2} \end{aligned}$$

g is diagonal

$g^{00} g_{00} = 1$

The perturbed field equations

Left-hand side (Einstein tensor):

$$G^0{}_0 = (-1 + 2\Psi) R_{00} - \frac{\mathcal{R}}{2}$$

To evaluate this, we need R_{00} (the Ricci tensor) and $\mathcal{R}$ (the Ricci scalar).

Since

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$$

we do need to calculate all the elements of *R:*

$$R_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha}$$

(Some of) the details will be done in an exercise.

The perturbed field equations

Result from exercise: the Christoffel symbols $\Gamma^0{}_{\mu\nu}$:

$$\Gamma^0{}_{00} = \Psi_{,0}$$

$$\Gamma^0{}_{i0} = \Gamma^0{}_{0i} = ik_i\tilde{\Psi}$$

$$\Gamma^0{}_{ij} = \delta_{ij}a^2\left[H + 2H(\Phi - \Psi) + \Phi_{,0}\right]$$

The remaining Christoffel symbols, $\Gamma^i{}_{\mu\nu}$:

$$\Gamma^i{}_{00} = \frac{ik^i}{a^2}\tilde{\Psi}$$

$$\Gamma^i{}_{j0} = \Gamma^i{}_{0j} = \delta_{ij}\left[H + \Phi_{,0}\right]$$

$$\Gamma^i{}_{jk} = i\tilde{\Phi}\left[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i\right]$$

The perturbed field equations

We can then calculate the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha}$$

The Ricci scalar can be conveniently separated into two components:

Zeroth order (unperturbed) component:

$$\mathcal{R}^{(0)} = 6\left(\frac{\mathrm{d}^2 a/\mathrm{d}t^2}{a} + H^2\right)$$

First order component:

$$\mathcal{R}^{(1)} = -12\Psi\left(H^2 + \frac{\mathrm{d}^2 a/\mathrm{d}t^2}{a}\right) + 2\Psi\frac{k^2}{a^2} + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\Phi\frac{k^2}{a^2}$$

The perturbed field equations

We now have everything we need for the Einstein tensor:

$$G^0{}_0 = (-1 + 2\Psi) R_{00} - \frac{\mathcal{R}}{2}$$

We will look only at the perturbed (first-order) part

$$\delta G^0{}_0 = (-1 + 2\Psi) \left[-\frac{(k)^2}{a^2} \Psi - 3 \frac{\mathrm{d}^2 a/\mathrm{d}t^2}{a} - 3\Phi_{,00} + 3(H\Psi_{,0} - 2H\Phi_{,0}) \right]$$

$$-\frac{1}{2} \left[-12\Psi \left(H^2 + \frac{\mathrm{d}^2 a/\mathrm{d}t^2}{a} \right) + 2\Psi \frac{k^2}{a^2} + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\Phi \frac{k^2}{a^2} \right]$$

Looks rather daunting, but simplifies to

$$\boxed{\delta G^0{}_0 = 6H^2\Psi - 6H\Phi_{,0} - 2\frac{k^2}{a^2}\Phi}$$

The perturbed field equations

We can now return to the Einstein equation: $G^0{}_0 = 8\pi G T^0{}_0$

$$\delta G^0{}_0 = 6H^2\Psi - 6H\Phi_{,0} - 2\frac{k^2}{a^2}\Phi$$

On the right-hand side, we have

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix}$$

with the perturbed part of $T^\mu{}_\nu$ being

$$\delta T^0{}_0 = -\left[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]$$

[for photons we have used that the energy density is ~ $T^4 = T_0(1+\Theta)^4 \sim T_0(1 + 4\Theta)$]

Putting it together

Including all species (DM, baryons, photons, neutrinos), we get

$$\delta T^0{}_0 = -\left[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]$$

Combining this with the left-hand side of the Einstein eqn ($\delta G^0{}_0$), we get

$$3H^2\Psi - 3H\Phi_{,0} - \frac{k^2}{a^2}\Phi = -4\pi G\left[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]$$

where the ‘tildes’ (~) have been dropped, but Ψ, Φ, and Θ refer to the respective Fourier components.

In terms of conformal time, with $\dot{a} \equiv {\rm d}a/{\rm d}\eta$

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\right) = 4\pi G a^2\left[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]$$

Note that for *a*=const, this reduces to the standard Poisson eqn.

Relating fluctuations in density, potential, and metric

One relation between the metric perturbations and the density:

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\right) = 4\pi G a^2 \left[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right]$$

Another relation can be obtained from the *spatial* components of the field eqn:

$$G^i{}_j = g^{ij}\left(R_{kj} - \frac{g_{kj}}{2}\mathcal{R}\right) = 8\pi G T^i{}_j$$

We do not go through the detailed derivation, but the resulting relation is

$$k^2(\Psi + \Phi) = -32\pi G a^2 \left[\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right]$$

for quadrupole moments Θ_2 and $\mathcal{N}_2$.

If the photon and neutrino perturbations have no quadrupole moments, then

$$\Psi = -\Phi$$

The Boltzmann equations

- **Photons:**

$$\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a \left[\Theta_0 - \Theta + \hat{p} \cdot \mathbf{v}_b \right]$$

- **(Cold) dark matter**: no collision terms; particles are non-relativistic.

Density fluctuations:

$$\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0$$

Velocity field:

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0$$

- **Baryons**: Collision terms from Coulomb scattering;

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0$$

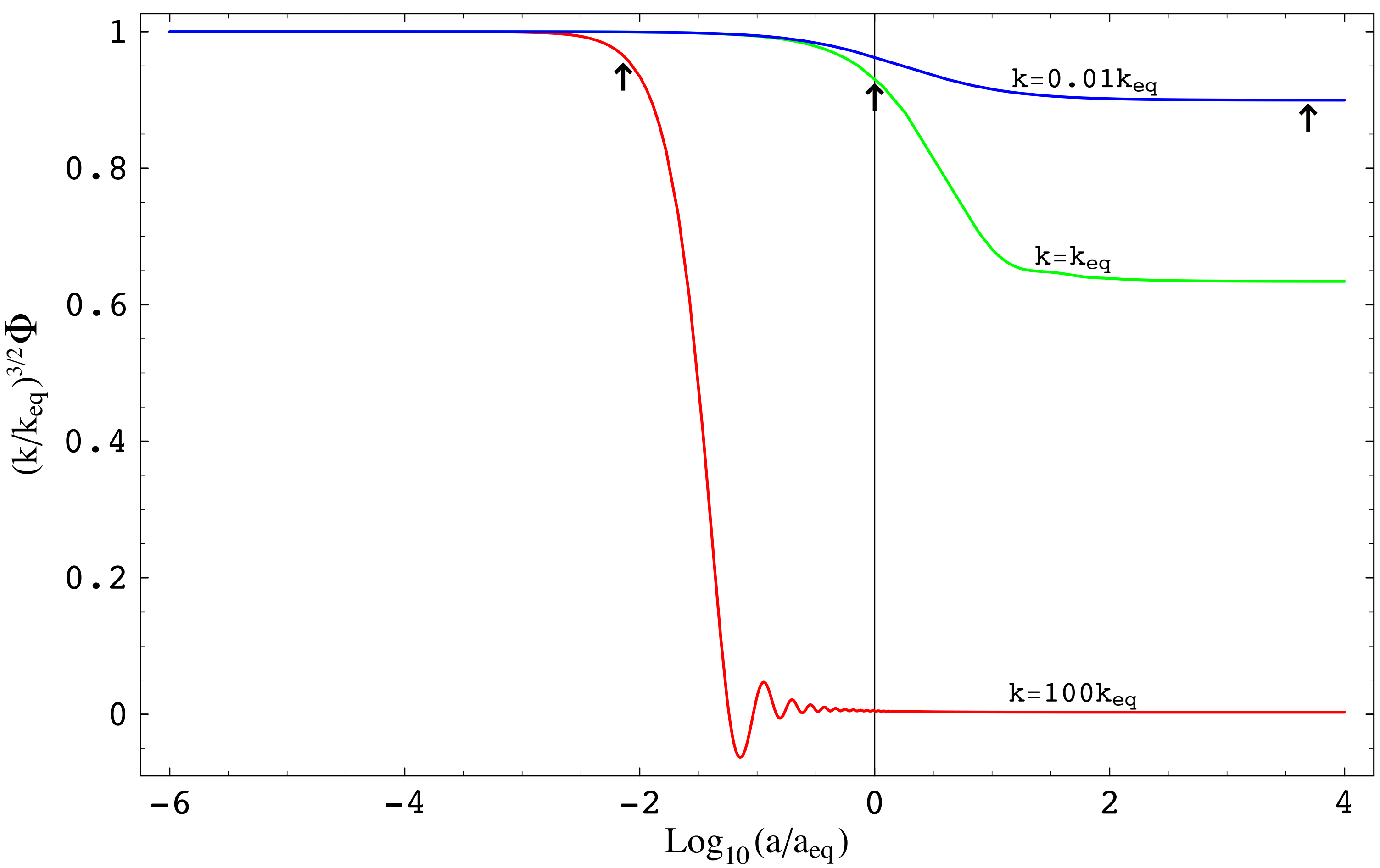
$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = n_e \sigma_T a \frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right]$$

- **Neutrinos**: similar to photons, but no collision terms

Evolution of density perturbations

- We have now derived:
  - The relativistic Boltzmann equations for the different constituents (dark/ordinary matter, photons, neutrinos)
  - Einstein equations for the potentials+curvature
- We now need to *solve* these equations.
- In general, this must be done *numerically*, using suitable initial conditions (e.g., as predicted by inflation)
- However, useful insight can be obtained analytically for certain specific situations

Evolution of the potential



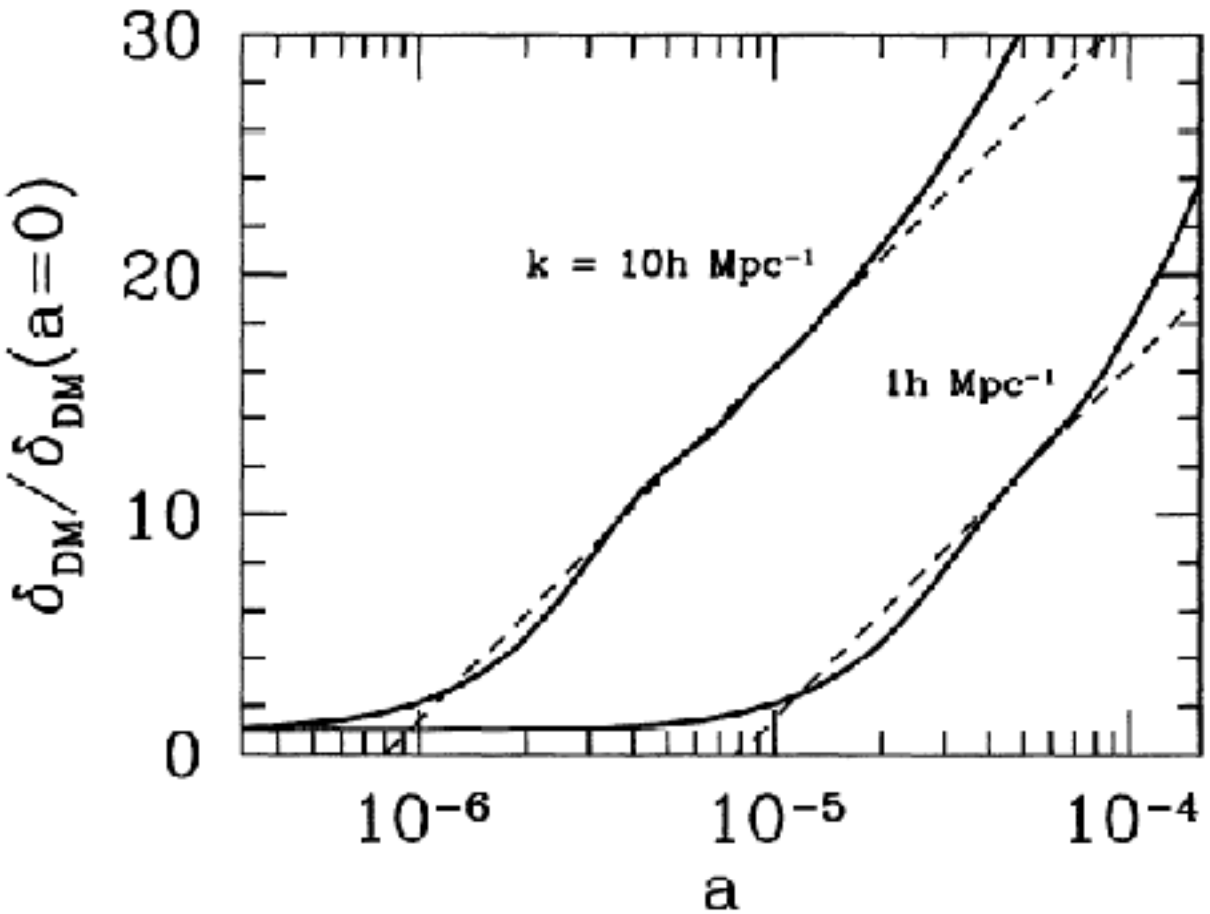
Early times:
- Fluctuations larger than horizon, potential does not evolve.

Intermediate times:
- Modes within horizon in radiation-dominated Universe
- Radiation pressure dominates
- Potential decays

Late times:
- Universe is matter dominated
- Potential is constant

Padmanabhan 2006
astro-ph/0602117

Growth of dark matter perturbations

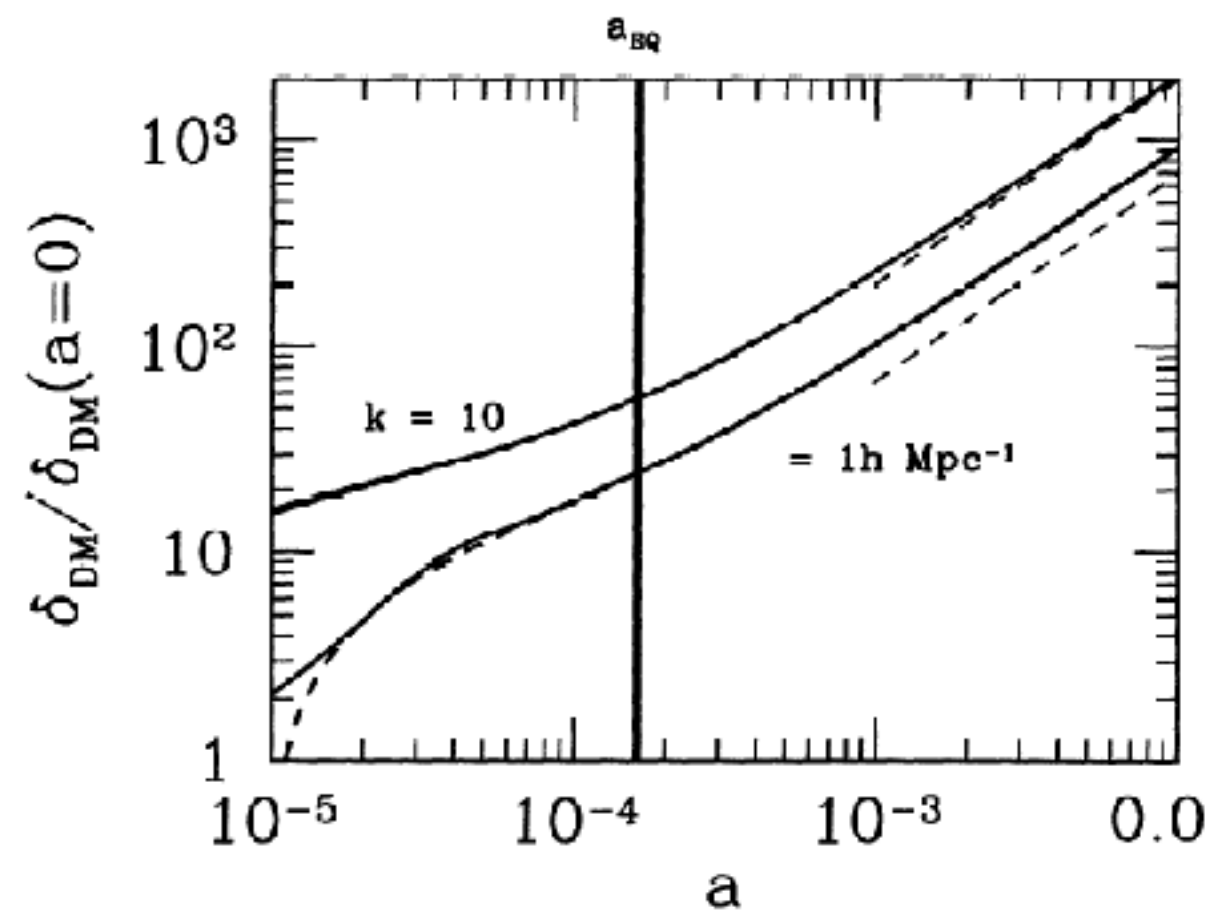


Radiation-dominated epoch:

- Growth slowed by decaying potentials, DM grows only *logarithmic.*

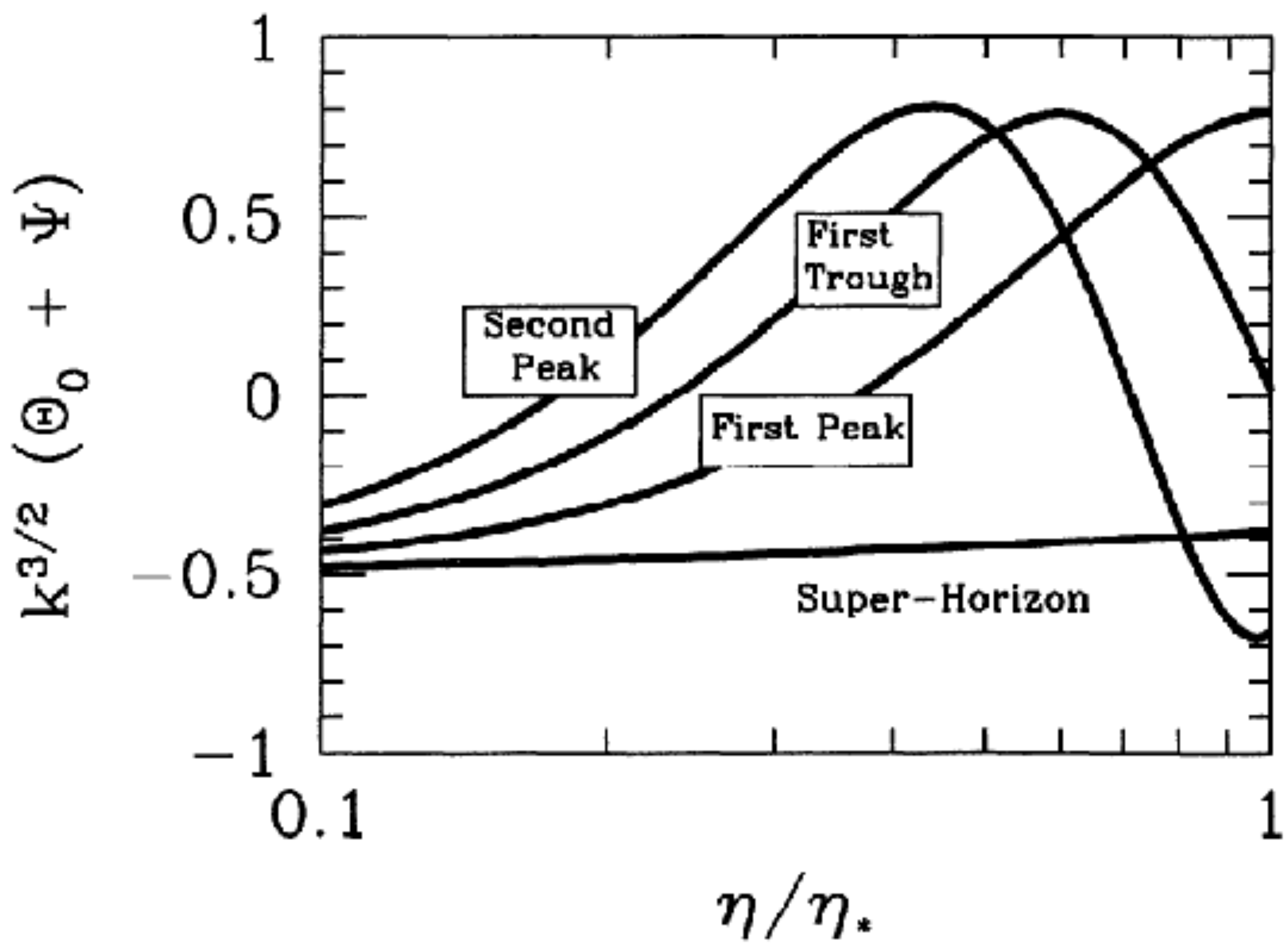
Matter-dominated epoch:

- Linear growth



Dodelson, *Modern Cosmology*.

Acoustic oscillations



Largest (super-horizon) scales:

Perturbations hardly evolve (no causal physics) - power-spectrum is “pristine” (i.e. as produced by inflation)

First peak:

Perturbations on this scale enter horizon at some epoch $\eta_1<\eta^*$ and just manage to reach maximum compression by η^*

First trough:

Perturbations enter horizon at $\eta_2<\eta_1$, “bounce”, and re-expand to average density by η^*

Second peak:

Perturbations re-expand to average rarefaction by η^*

What about the spring analogies?

Acoustic oscillations

So much for the general picture. Now let's look at the details!

The tightly coupled limit:

Before recombination (η*), mean free path for a photon was much smaller than horizon.

Define *optical depth* as integral of $n_e \sigma_T a$ over (conformal) time:

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} \mathrm{d}\eta' n_e \sigma_T a$$

with derivative:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\eta} \equiv \dot{\tau} = -n_e \sigma_T a$$

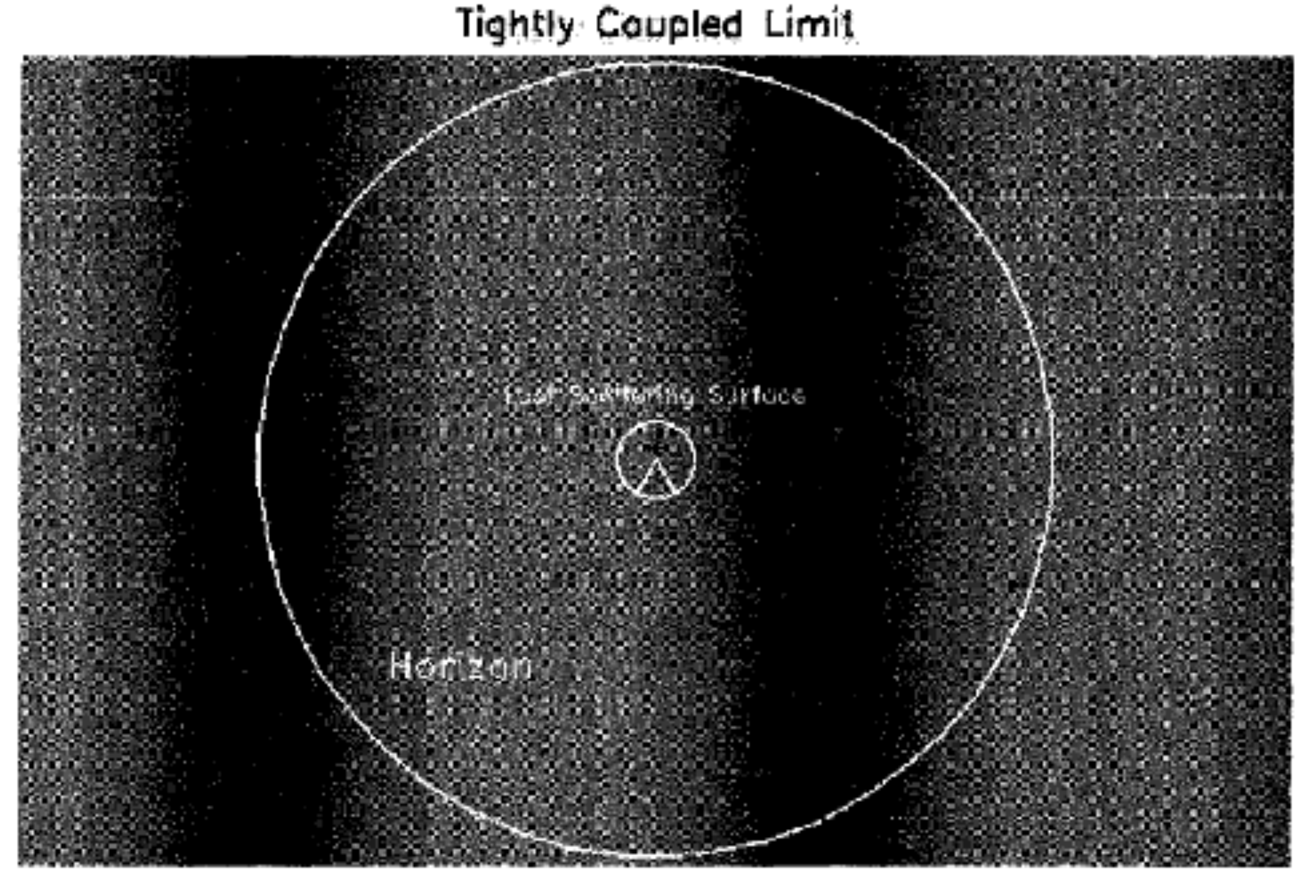
Tightly coupled limit corresponds to τ >> 1.

Acoustic oscillations

The tightly coupled limit:

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} \mathrm{d}\eta' n_e \sigma_T a$$

Tightly coupled limit corresponds to $\tau >> 1$ (last scattering surface much smaller than horizon)



Higher-order moments of radiation field are then negligible: Θ “looks the same in every direction”, apart from spatial and velocity dependencies. We only need to consider
[Θ_0(**x**, t)] - Monopole
[Θ_1(**x**, t)] - Dipole

(see Sect. 8.3.1 for formal derivation).

Multipole moments

We define the *l*th multipole moment of Θ as

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mathcal{P}_l(\mu) \Theta(\mu)$$

The first three Legendre polynomials are defined as:

$$\mathcal{P}_0(\mu) = 1$$

$$\mathcal{P}_1(\mu) = \mu$$

$$\mathcal{P}_2(\mu) = \frac{3\mu^2 - 1}{2}$$

Acoustic oscillations

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\mathcal{P}_l(\mu)\Theta(\mu)$$

$$\mathcal{P}_0(\mu) = 1 \qquad \mathcal{P}_1(\mu) = \mu$$

The Boltzmann equation for photons in the tightly coupled limit:

General version:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + n_e\sigma_T a\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]$$

Next: obtain two new equations by multiplying by P_0 and P_1 and integrating over all μ, dropping higher-order moments.

P_0, left-hand side:

$$\begin{aligned}\int \mathrm{d}\mu\,\dot{\Theta} + ik\mu\Theta &= 2\dot{\Theta}_0 + ik\int_{-1}^{1}\mathrm{d}\mu\,\mu\Theta(\mu)\\ &= 2\dot{\Theta}_0 + ik(-2i)\Theta_1\\ &= 2\dot{\Theta}_0 + 2k\Theta_1\end{aligned}$$

Acoustic oscillations

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{\mathrm{d}\mu}{2}\mathcal{P}_l(\mu)\Theta(\mu)$$

$$\mathcal{P}_0(\mu) = 1 \qquad \mathcal{P}_1(\mu) = \mu$$

The Boltzmann equation for photons in the tightly coupled limit:

General version:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + n_e\sigma_T a\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]$$

P_0, left-hand side: $= 2\dot{\Theta}_0 + 2k\Theta_1$

P_0, right-hand side:

$$\int_{-1}^{1}\mathrm{d}\mu\left(-\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]\right) = -\int_{-1}^{1}\mathrm{d}\mu\dot{\Phi} - ik\int_{-1}^{1}\mathrm{d}\mu\mu\Psi - \dot{\tau}\left[\int_{-1}^{1}\mathrm{d}\mu(\Theta_0 - \Theta) + v_b\int_{-1}^{1}\mathrm{d}\mu\mu\right]$$

$-2\dot{\Phi}$ $\quad 0 \quad$ $-2\dot{\tau}\Theta_0$ $\quad 2\dot{\tau}\Theta_0 \quad$ 0

Equating l.h. and r.h. sides:

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi}$$

Acoustic oscillations

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mathcal{P}_l(\mu)\Theta(\mu)$$

$$\mathcal{P}_0(\mu) = 1 \qquad \mathcal{P}_1(\mu) = \mu$$

The Boltzmann equation for photons in the tightly coupled limit:

General version:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + n_e\sigma_T a\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]$$

P$_1$, left-hand side:

$$\int_{-1}^{1} \mathrm{d}\mu\, \mu \left(\dot{\Theta} + ik\mu\Theta\right) = -2i\dot{\Theta}_1 + ik\int_{-1}^{1} \mathrm{d}\mu\mu^2\Theta$$

$$= -2i\dot{\Theta}_1 + 2ik\left(\frac{1}{3}\Theta_0 - \frac{2}{3}\Theta_2\right)$$

Exercise

Acoustic oscillations

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{\mathrm{d}\mu}{2} \mathcal{P}_l(\mu)\Theta(\mu)$$

$$\mathcal{P}_0(\mu) = 1 \qquad \mathcal{P}_1(\mu) = \mu$$

The Boltzmann equation for photons in the tightly coupled limit:

General version:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + n_e\sigma_T a\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]$$

P_1, right-hand side:

$$\int_{-1}^{1} \mathrm{d}\mu\,\mu\left(-\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]\right) =$$

$$-\int_{-1}^{1}\mathrm{d}\mu\mu\dot{\Phi} - ik\int_{-1}^{1}\mathrm{d}\mu\mu^2\Psi - \dot{\tau}\left[\int_{-1}^{1}\mathrm{d}\mu\mu(\Theta_0 - \Theta) + v_b\int_{-1}^{1}\mathrm{d}\mu\mu^2\right]$$

$$0 \qquad -\frac{2}{3}ik\Psi \qquad 0 \qquad -2\dot{\tau}i\Theta_1 \qquad -\frac{2}{3}\dot{\tau}v_b$$

$$= -\frac{2}{3}ik\Psi - \dot{\tau}\left[2i\Theta_1 + \frac{2}{3}v_b\right]$$

Acoustic oscillations

The Boltzmann equation for photons in the tightly coupled limit:

General version:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi + n_e\sigma_T a\left[\Theta_0 - \Theta + \mu\mathbf{v}_b\right]$$

Equating l.h. and r.h. sides:

$$-2i\dot{\Theta}_1 + 2ik\left(\frac{1}{3}\Theta_0 - \frac{2}{3}\Theta_2\right) = -\frac{2}{3}ik\Psi - \dot{\tau}\left[2i\Theta_1 + \frac{2}{3}v_b\right]$$

Dividing by 2*i* and dropping the Θ_2 term:

$$\boxed{\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right]}$$

Acoustic oscillations

The Boltzmann equation for photons in the tightly coupled limit:

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi}$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right]$$

Two equations for Θ_0 and Θ_1 and their derivatives.
We would like to have a single equation for each multipole (and eliminate v_b).

For v_b, invoke the B.E. for baryons:

The Boltzmann equations

- **Photons:**

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = n_e\sigma_T a\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\mathbf{v}_b\right]$$

- **(Cold) dark matter**: no collision terms; particles are non-relativistic.

Density fluctuations:

$$\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0$$

Velocity field:

$$\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0$$

- **Baryons**: Collision terms from Coulomb scattering;

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0$$

$$\boxed{\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = n_e\sigma_T a\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right]}$$

- **Neutrinos**: similar to photons, but no collision terms