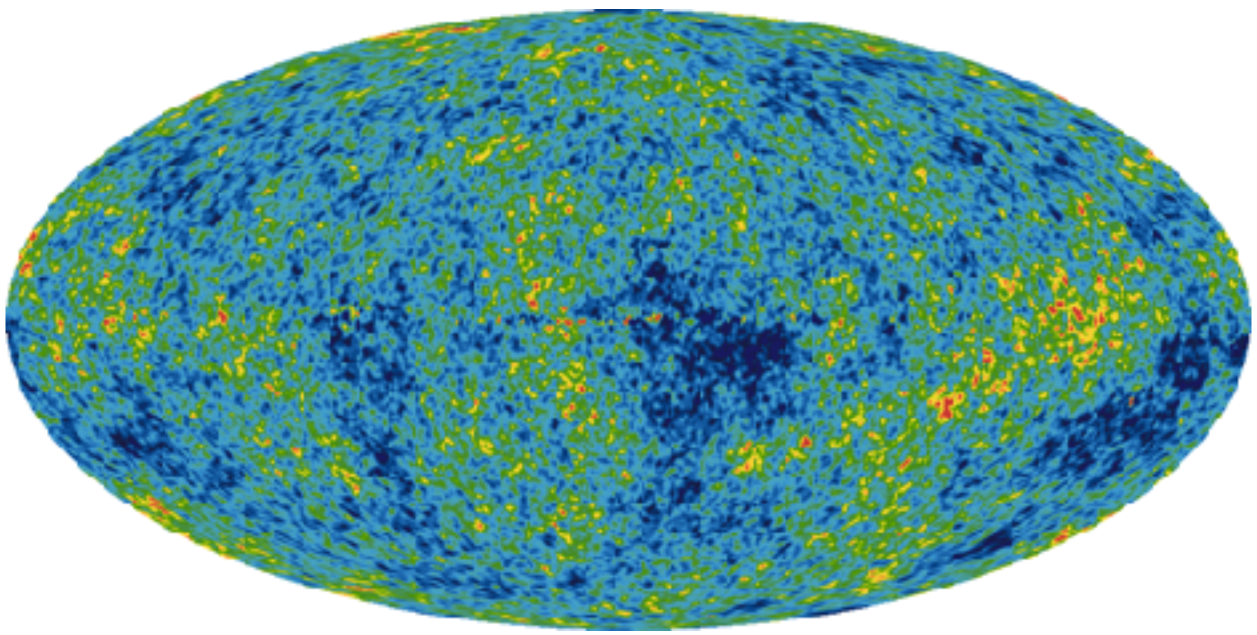
340th anniversary of Ole Rømer's determination of the speed of light.


L'ASTRONOME DANOIS
OLAUS RÖMER
1644 - 1710
A DECOUVERT LA VITESSE DE PROPAGATION
DE LA LUMIÈRE
A L'OBSERVATOIRE DE PARIS
EN 1676

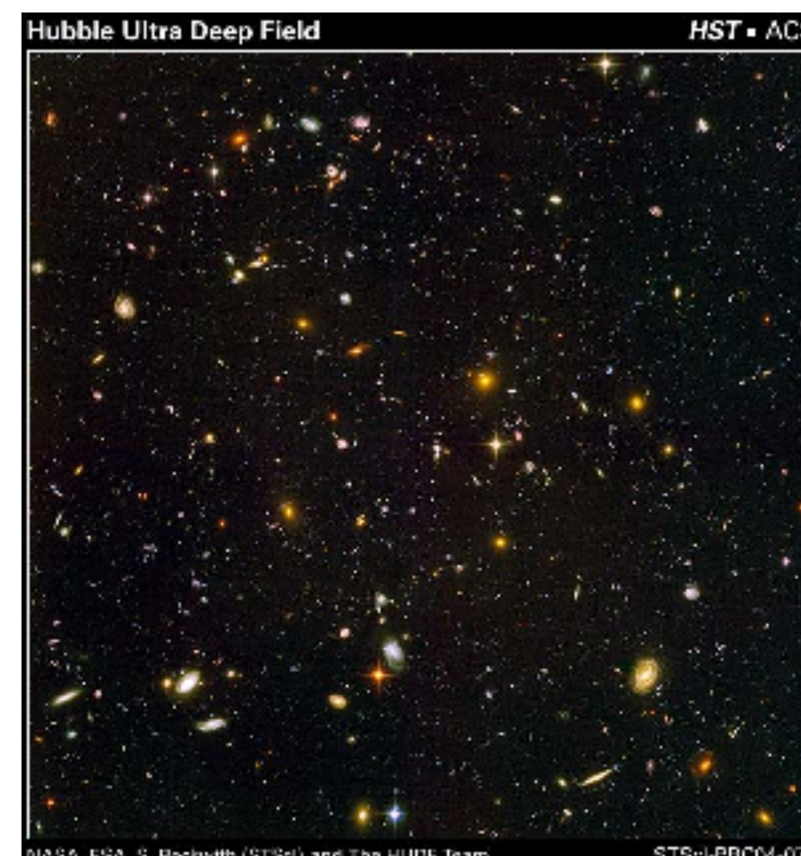
The Development and Growth of Density Fluctuations

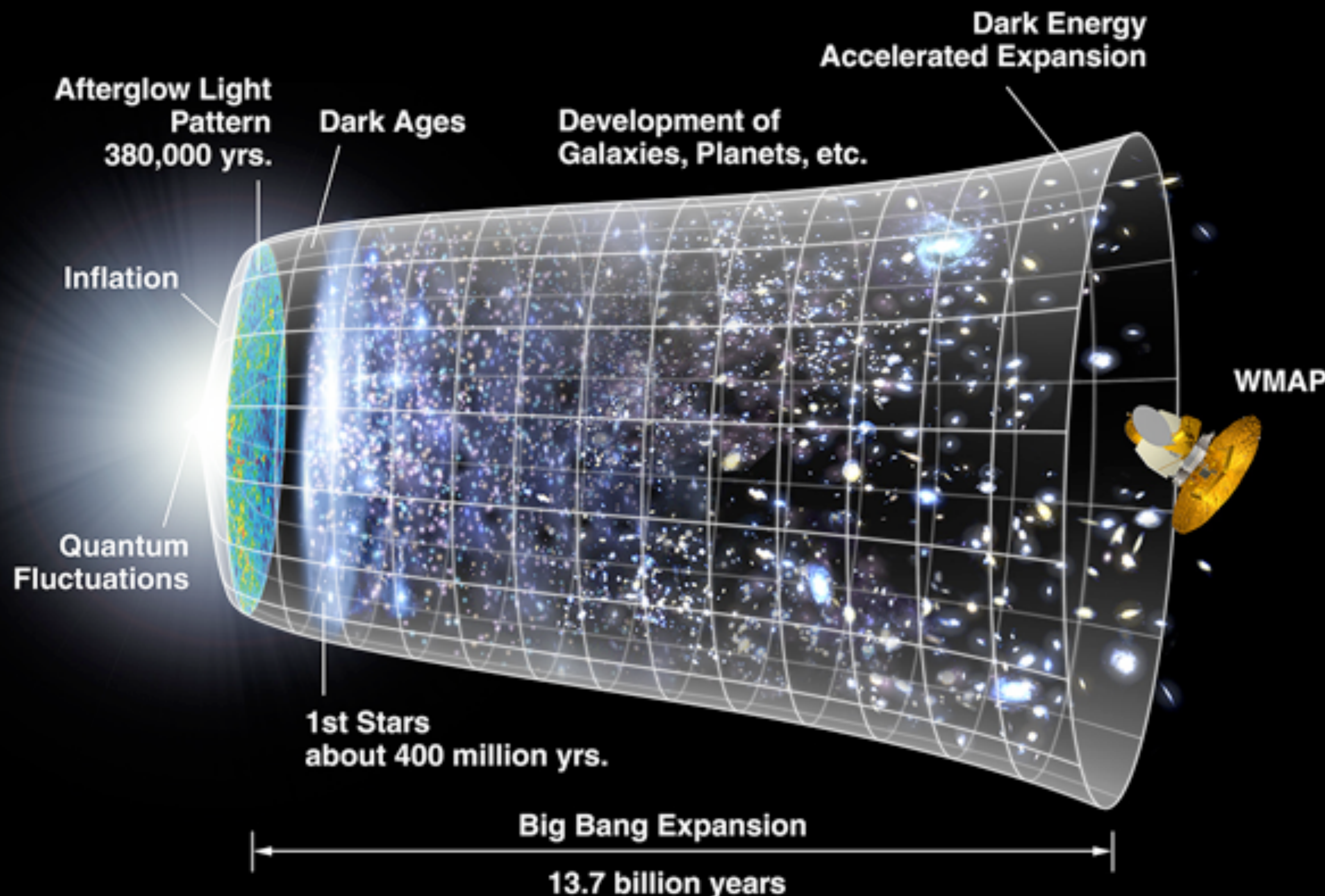
The basic problem

How to get from here:



to here:





NASA/WMAP Scienc

Present-day densities

Mean density of Universe:

$$\rho_0 \approx 0.3\rho_{\rm crit} = 0.3\frac{3H_0^2}{8\pi G} \approx 3\times 10^{-27}\,{\rm kg\,m^{-3}}$$

Mean density of galaxy clusters:

$$\rho \approx \left(\frac{10^{15}\,M_\odot}{\frac{4}{3}\pi(1\,{\rm Mpc}^3)}\right) \approx 2\times 10^{-23}\,{\rm kg\,m^{-3}} \quad \approx 5000\,\rho_0$$

Mean density of galaxies:

$$\rho_{\rm MW} \approx \left(\frac{4\times 10^{11}\,M_\odot}{\frac{4}{3}\pi(20\,{\rm kpc}^3)}\right) \approx 10^{-21}\,{\rm kg\,m^{-3}} \quad \approx 3\times 10^{5}\,\rho_0$$

Present-day densities

Present-day overdensities of galaxies and galaxy clusters are

$$\Delta = \delta\rho/\rho_0 \approx 10^3 \qquad \text{(clusters)}$$

$$\Delta = \delta\rho/\rho_0 \approx 10^5 \qquad \text{(galaxies)}$$

Mean density of Universe:

$$\rho \propto \rho_0(1+z)^3$$

As virialized objects, galaxies and clusters must have segregated out *after* z~50 and z~10, respectively.

Accessible to observations! (at least in principle).

Density contrast small ($\Delta \ll 1$) at higher redshifts - linear regime!

Growth of perturbations

- Important distinction between Dark and Baryonic matter.
- DM: Only gravity - relatively “easy”, especially at early epochs when fluctuations are still small
- Baryonic matter: Complicated! - not just gravity, but also dissipational processes, feedback, heating/cooling, etc.

Fate of overdense regions

Static case (e.g. molecular cloud): overdensity collapses on a free-fall time scale,

$$t_{\mathrm{ff}} = \left(\frac{3\pi}{32\, G\, \rho} \right)^{1/2}$$

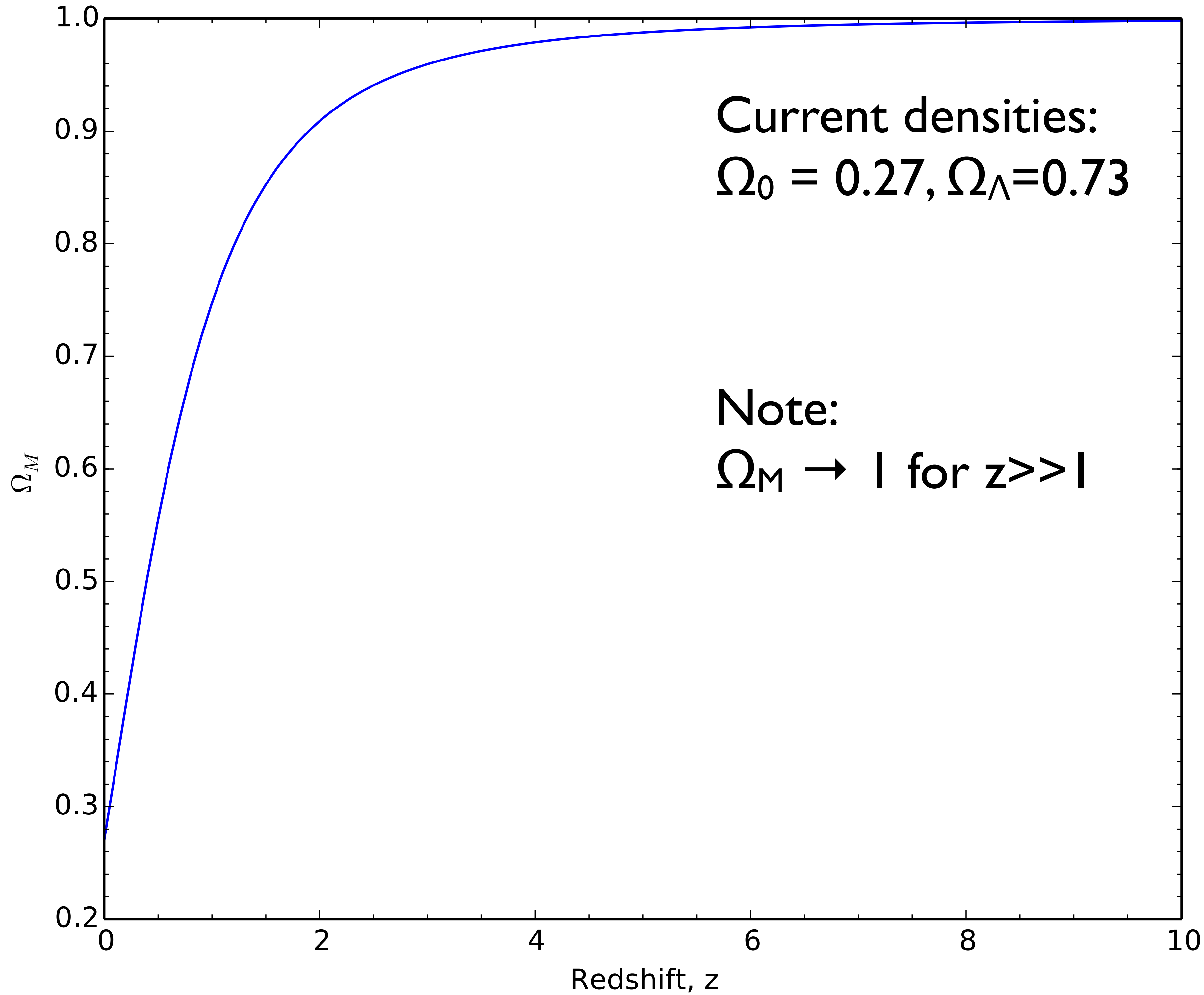
Expanding Universe:

Think of over-dense regions as "mini-Universes" of slightly higher density than Ω_0 in a critical (Einstein-de Sitter) Universe.

"Background" scale factor (for Ω=1)

$$a(t) = \left(\frac{3 H_0 t}{2} \right)^{2/3}$$

Overdense region with $\Omega' > 1$ will eventually reach maximum *a'(t)* and then re-collapse.


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
Ω_M
0
2
4
6
8
10
Redshift, z
Current densities:
Ω_0 = 0.27, Ω_Λ=0.73
Note:
$\Omega_M \rightarrow 1$ for z>>1

Fate of overdense regions

"Background" scale factor:

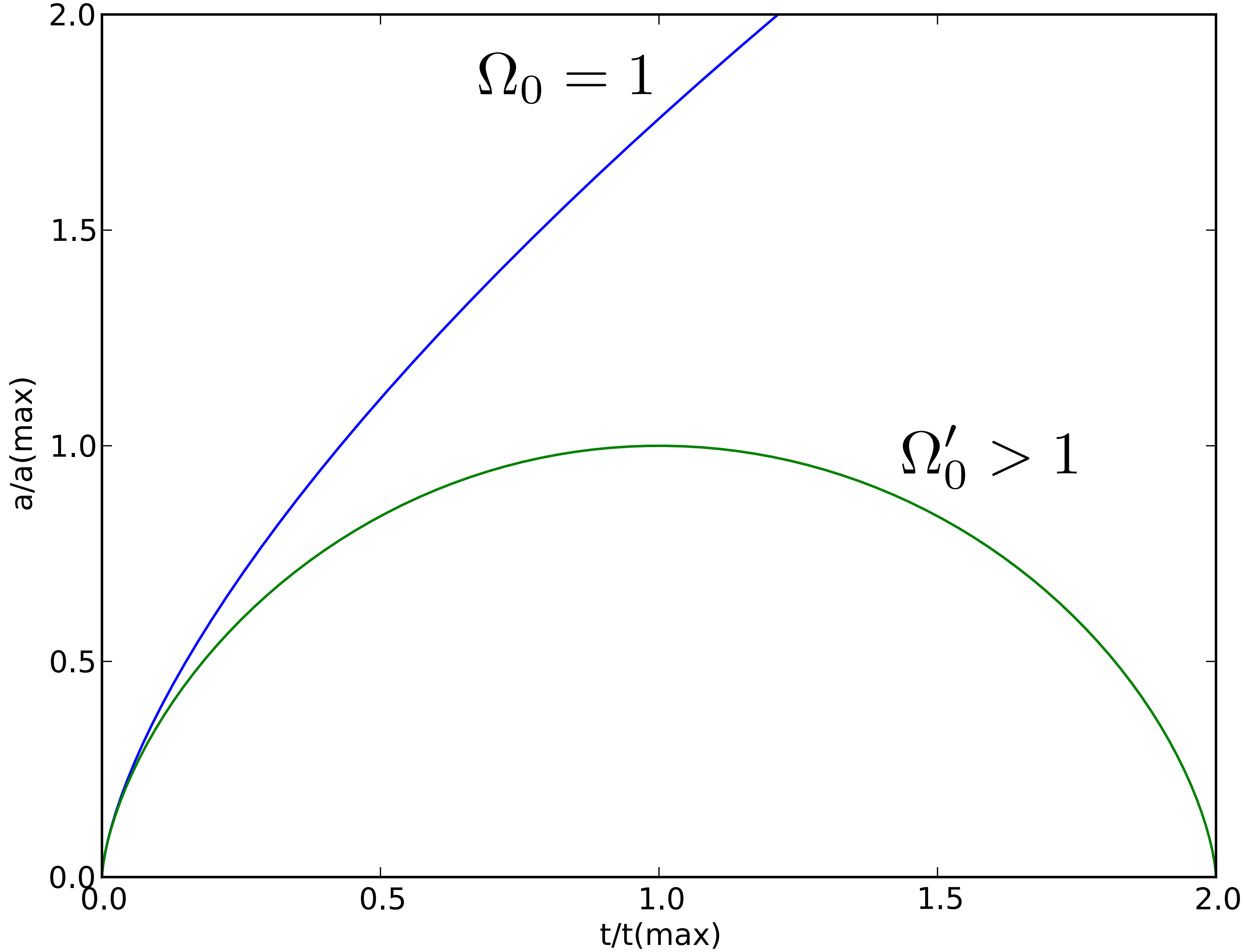
$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3}$$

Evolution of over-dense region:

Consider the parametric solutions to the Friedmann equation for Ω_Λ=0 and Ω_0 >1 (assignment):

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta)$$

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$



$\Omega_0 = 1$
$\Omega_0' > 1$
a/a(max)
t/t(max)
0.0
0.5
1.0
1.5
2.0

Fate of over-dense regions

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta)$$

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

To find approximate relation for *a(t)*, Taylor-expand the expressions for *a*(θ) and *t*(θ) around θ=0:

$$\cos\theta \approx 1 + \frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}\theta + \frac{1}{2}\frac{\mathrm{d}^2\cos\theta}{\mathrm{d}\theta^2}\theta^2 + \frac{1}{6}\frac{\mathrm{d}^3\cos\theta}{\mathrm{d}\theta^3}\theta^3 + ...$$

$$\approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$$

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3$$

Fate of overdense regions

Solving for the scale factor *a'(t)* of the perturbation, we get:

$$a' \simeq (\Omega_0')^{1/3} \left(\frac{3H_0 t}{2}\right)^{2/3} \left[1 - \frac{1}{12}\left(\frac{12tH_0(\Omega_0' - 1)^{3/2}}{\Omega_0'}\right)^{2/3}\right]$$

$$\underbrace{(\Omega_0')^{1/3}}_{\Omega_0' \approx \Omega_0} \quad \underbrace{\left(\frac{3H_0 t}{2}\right)^{2/3}}_{a(t)} \quad \underbrace{\left[\ldots\right]}_{\text{Growth of perturbation}}$$

Fate of overdense regions

$$a' \simeq (\Omega_0')^{1/3}\left(\frac{3H_0 t}{2}\right)^{2/3}\left[1-\frac{1}{12}\left(\frac{12tH_0(\Omega_0'-1)^{3/2}}{\Omega_0'}\right)^{2/3}\right]$$

The density of the fluctuation is then

$$\rho' = \rho_0'(a')^{-3}$$

Using $(1-\delta)^{-3} \approx 1+3\delta$ for $\delta \ll 1$:

$$\rho' \simeq \rho_0'/\Omega_0' a^{-3}\left[1+\frac{3}{12}\left(\frac{12tH_0(\Omega_0'-1)^{3/2}}{\Omega_0'}\right)^{2/3}\right]$$

$$\rho' \simeq \rho_0 a^{-3}\left[1+a\left(\frac{\Omega_0'-1}{(\Omega_0')^{2/3}}\right)\right]$$

Fate of overdense regions

$$\rho' \simeq \rho_0 a^{-3}\left[1 + a\left(\frac{\Omega_0' - 1}{(\Omega_0')^{2/3}}\right)\right]$$

The density contrast then grows as

$$\Delta = \frac{\delta\rho}{\rho} = \frac{\rho' - \rho}{\rho} \simeq \frac{\rho_0 a^{-3}\left[1 + a\left(\frac{\Omega_0' - 1}{(\Omega_0')^{2/3}}\right)\right] - \rho_0 a^{-3}}{\rho_0 a^{-3}}$$

$$\Delta = a\left(\frac{\Omega_0' - 1}{(\Omega_0')^{2/3}}\right)$$

Key Point:

Small density contrasts Δ grow *slowly* with *a* (approximately linearly):

$$\Delta = \left(\frac{\Omega_0' - 1}{(\Omega_0')^{2/3}} \right) a$$

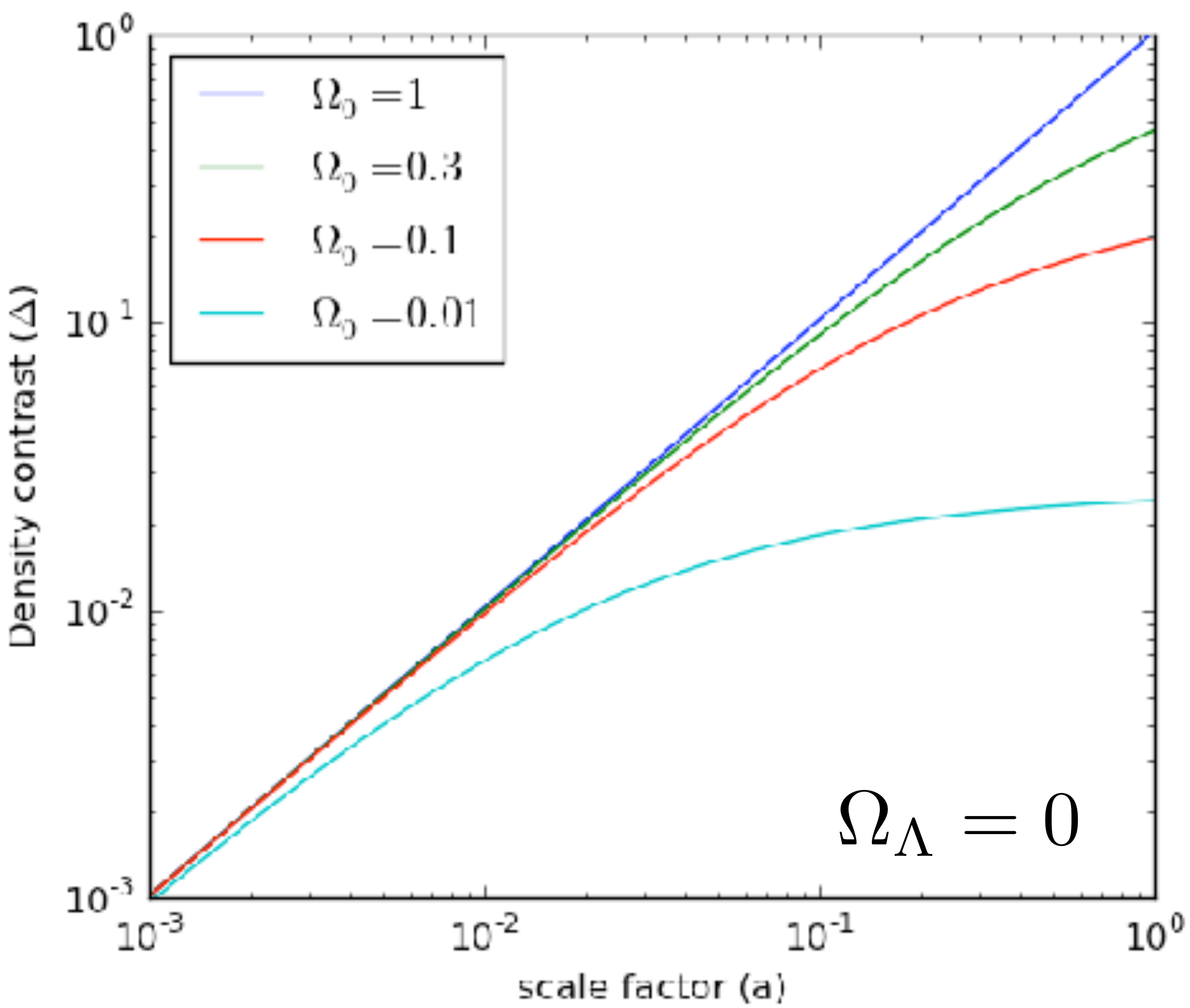
Derived here for the special case of an Einstein-de Sitter Universe, but (qualitatively) true generally.

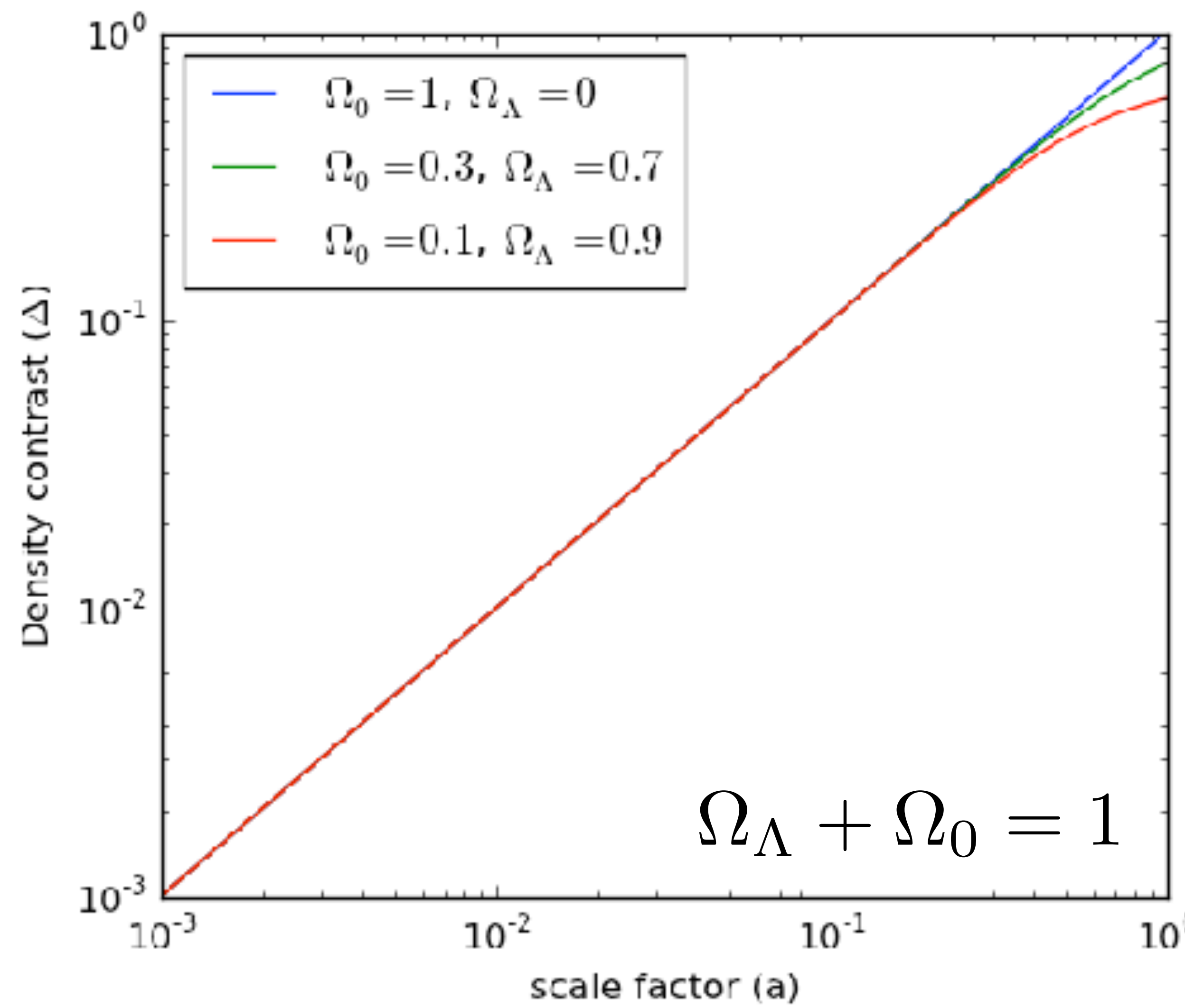
Growth of small fluctuations

More general analysis
(still assuming no pressure):

$$\Delta(a) = \frac{5\Omega_0}{2}\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)\int_0^a \left(\frac{\mathrm{d}a'}{\mathrm{d}t}\right)^{-3}\mathrm{d}a'$$

Heath (1977); Carroll et al. (1992)





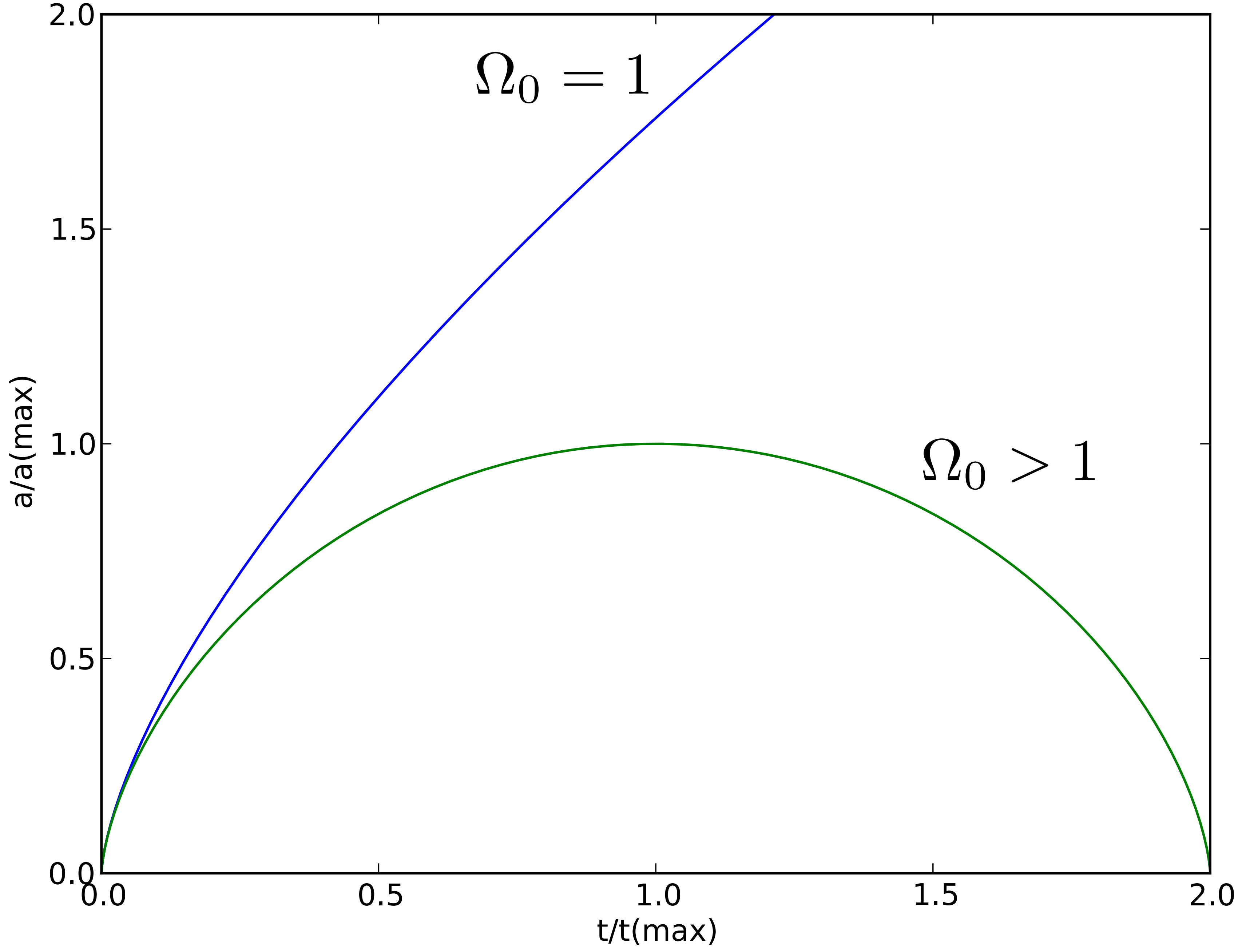
The linear regime

- Fluctuations that are now in the non-linear regime ($\Delta \gtrsim 1$) must already have had significant amplitudes at recombination.
- At z=1000, present-day virialized structures (galaxies, clusters) must have corresponded to $\Delta \gtrsim 10^{-3}$ - much larger than the fluctuations seen in the *baryonic* matter in the CMB ($\sim 10^{-5}$)!

When did virialized structures form?

- "Top-hat" model:
  Consider collapse of spherically symmetric over-densities.
- Initially: evolve as "mini-Universes" with higher than critical density
- At some time t_{max} these reach maximum scale factor a_{max}, and then recollapse

Top-hat model



The top-hat model

Follow the evolution of a "mini-Universe" with $\Omega_0 > 1$:

$$a(\theta) = \frac{\Omega_0'}{2(\Omega_0' - 1)}(1 - \cos\theta)$$

$$t(\theta) = \frac{\Omega_0'}{2H_0(\Omega_0' - 1)^{3/2}}(\theta - \sin\theta)$$

Will reach maximum scale factor a_{max} for $\theta = \pi$, and recollapse for $\theta = 2\pi$.
At the "turn-around" point ($\cos\theta = -1$),

$$a_{\max} = \frac{\Omega_0'}{\Omega_0' - 1} \qquad t_{\max} = \frac{\pi\Omega_0'}{2H_0(\Omega_0' - 1)^{3/2}}$$

Top-hat model

At the "turn-around" point,

$$a_{\max} = \frac{\Omega_0'}{\Omega_0' - 1} \qquad t_{\max} = \frac{\pi \Omega_0'}{2H_0(\Omega_0' - 1)^{3/2}}$$

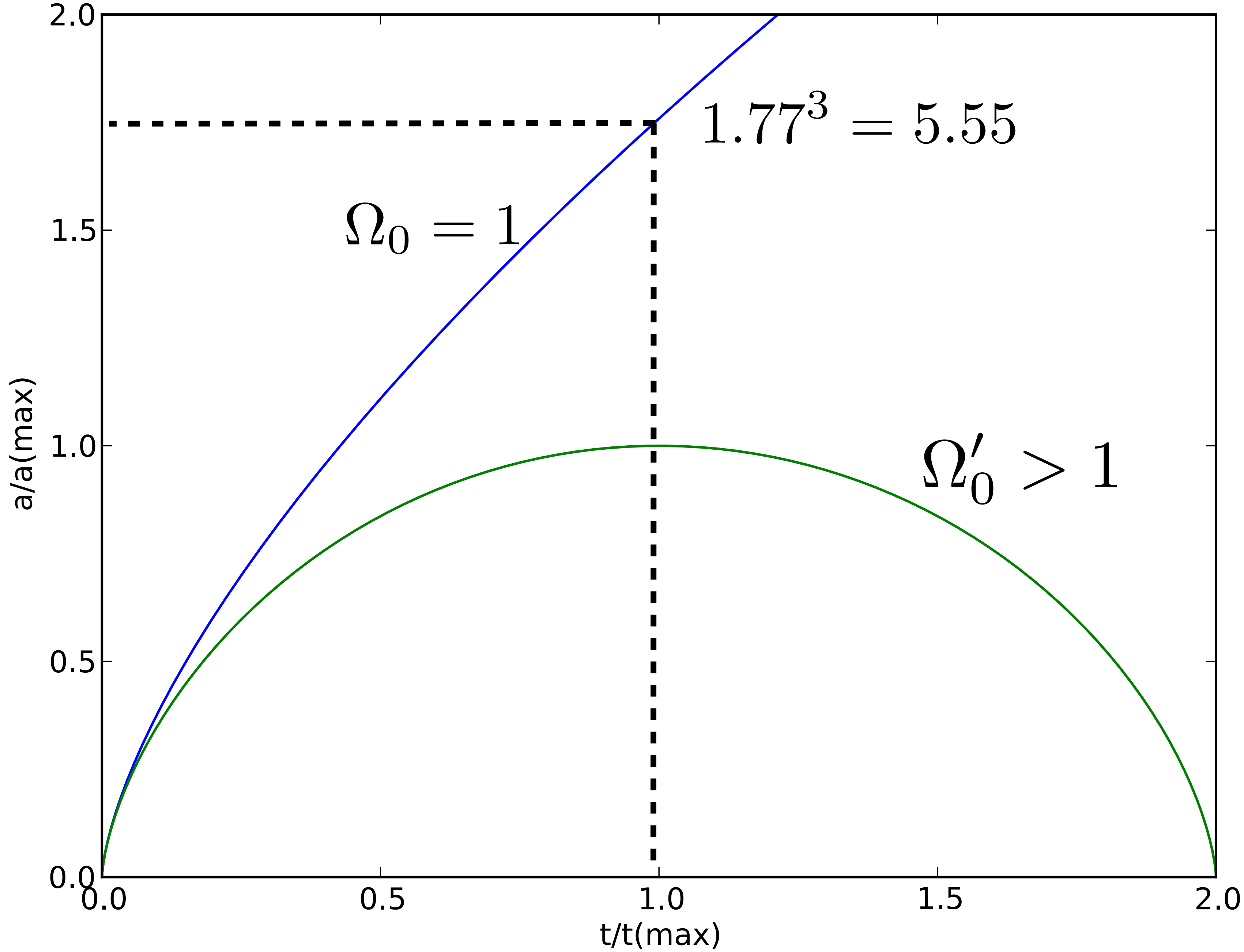
while the background scale factor is

$$a = \left(\frac{3H_0 t_{\max}}{2}\right)^{2/3}$$

The density contrast is then

$$\frac{\rho_{\max}}{\rho_0} = \Omega_0' \left(\frac{a}{a_{\max}}\right)^3 \qquad = \Omega_0' \left\{ \frac{\left(\frac{3H_0}{2} \frac{\pi \Omega_0'}{2H_0(\Omega_0'-1)^{3/2}}\right)}{\frac{\Omega_0'}{\Omega_0'-1}} \right\}^3$$

$$= 9\pi^2/16 \approx 5.55$$



2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
a/a(max)
t/t(max)
$1.77^3 = 5.55$
$\Omega_0 = 1$
$\Omega_0' > 1$

Top-hat model cont'd

By the time the perturbation has decoupled completely (expansion halted), it already has $\Delta \approx 5$.

"Total recollapse" occurs at

$$t_{\rm coll} = 2\, t_{\rm max}$$

In terms of redshift,

$$\frac{1+z_{\rm max}}{1+z_{\rm coll}} = \frac{a_{\rm max}}{a_{\rm coll}} = \left(\frac{t_{\rm coll}}{t_{\rm max}}\right)^{2/3}$$

$$1+z_{\rm coll} = \frac{1+z_{\rm max}}{2^{2/3}}$$

E.g, for z_{coll}=0 (now), we get z_{max}~0.6

The collapse

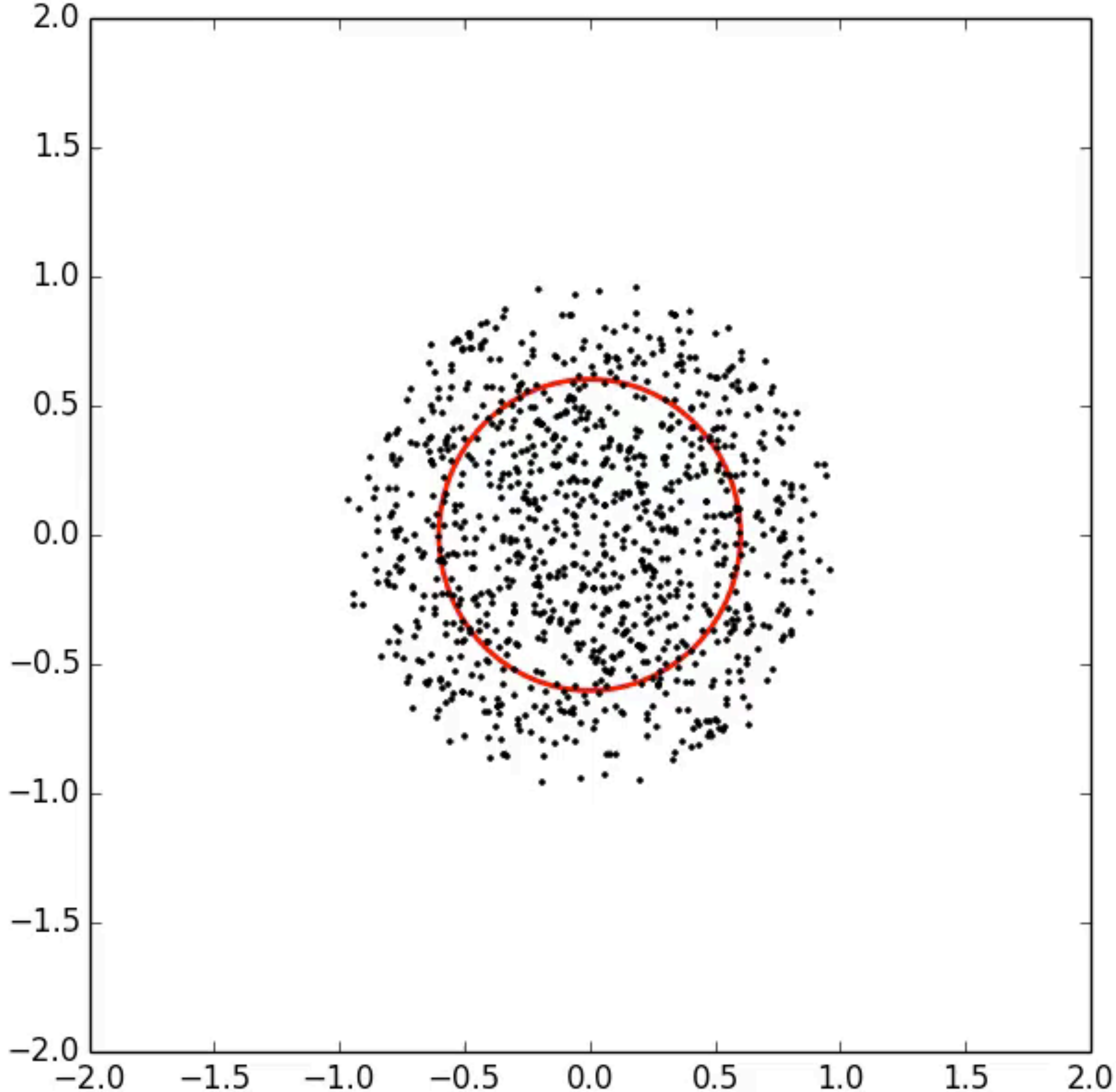
Overdensity reaches virial equilibrium on "violent relaxation" time scale:

$$T_r \approx \frac{3P}{8\pi}$$

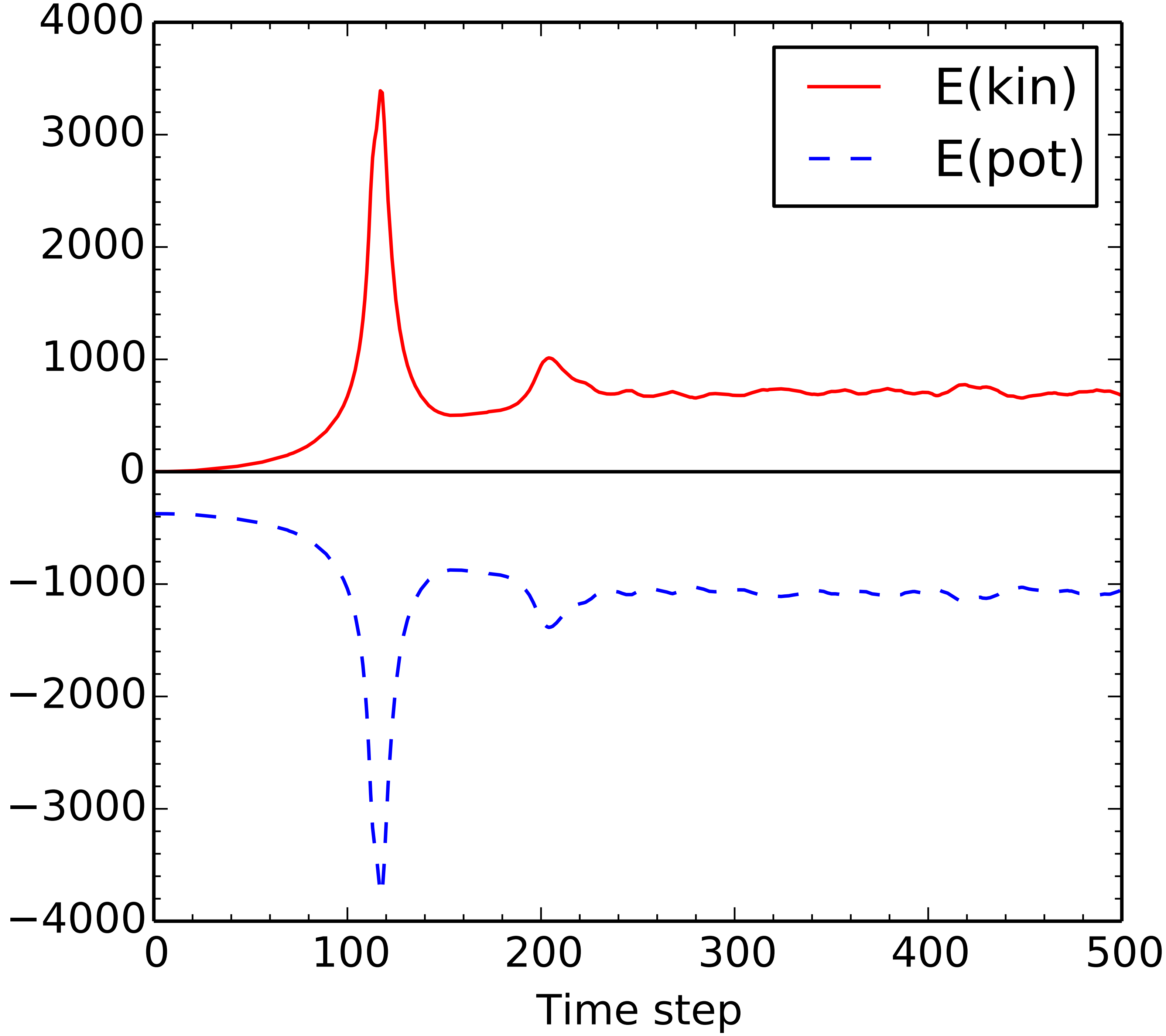
where *P* is roughly the crossing time of the system (Lynden-Bell 1967).

Applies to systems that are initially far from equilibrium configuration.

Solution to "Zwicky's paradox" (Zwicky 1939):
Two-body relaxation time scales of galaxy clusters are of order 10^{18} years, why do they appear as symmetric as they do?



2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0



E(kin)
E(pot)
4000
3000
2000
1000
0
−1000
−2000
−3000
−4000
0
100
200
300
400
500
Time step

The collapse

At point of maximum expansion, t_{max}:

Perturbation of mass *M* has some radius r_{max}.

Potential energy:

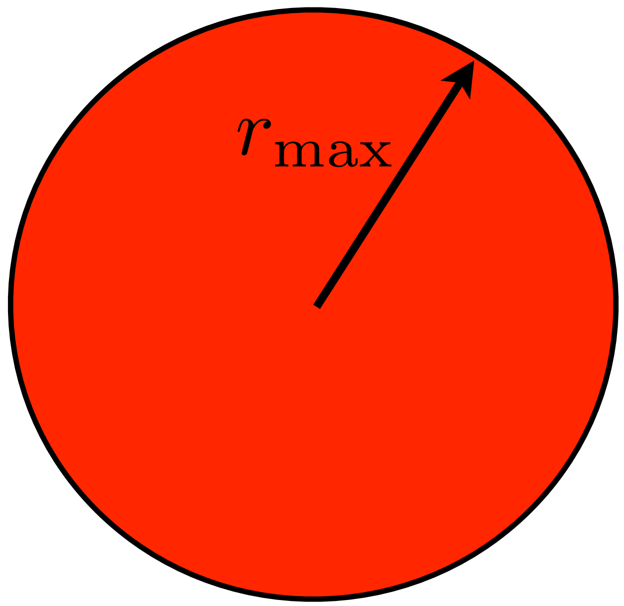
$$U = -\frac{3}{5}\frac{GM^2}{r_{max}}$$

Kinetic energy at this point:

$$T = 0$$

$$U = U(r_{\mathrm{max}})$$

$$T = 0$$



The collapse

Virial equilibrium is reached when

$$T = -\frac{1}{2}U$$

By energy conservation,

$$T(r_{\rm vir}) = U(r_{\rm max}) - U(r_{\rm vir})$$

that is,

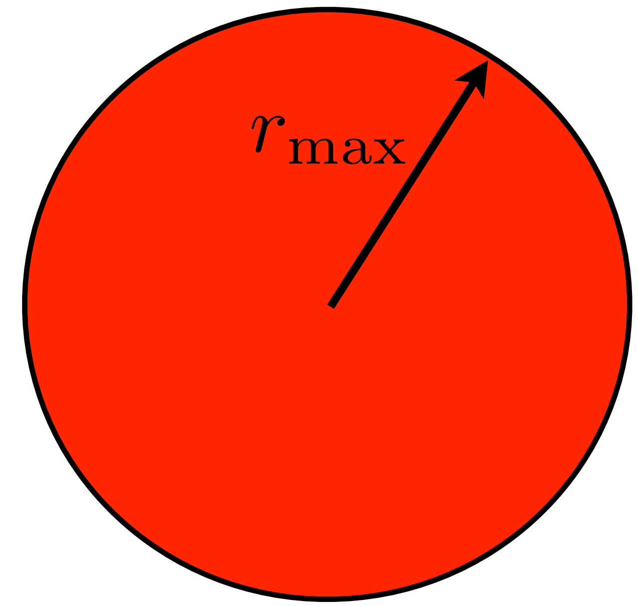
$$U(r_{\rm max}) = \frac{1}{2}U(r_{\rm vir})$$

$$-\frac{3}{5}\frac{GM^2}{r_{\rm max}} = -\frac{1}{2}\frac{3}{5}\frac{GM^2}{r_{\rm vir}}$$

$$r_{\rm vir} = \frac{1}{2}r_{\rm max}$$

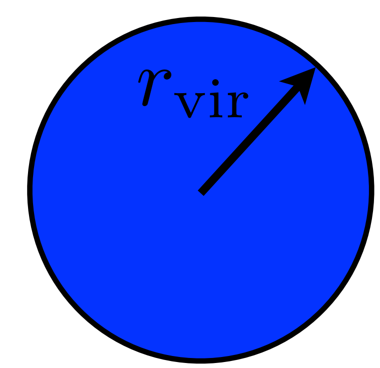
Maximum expansion:

$$T = 0 \quad U = U(r_{\rm max})$$

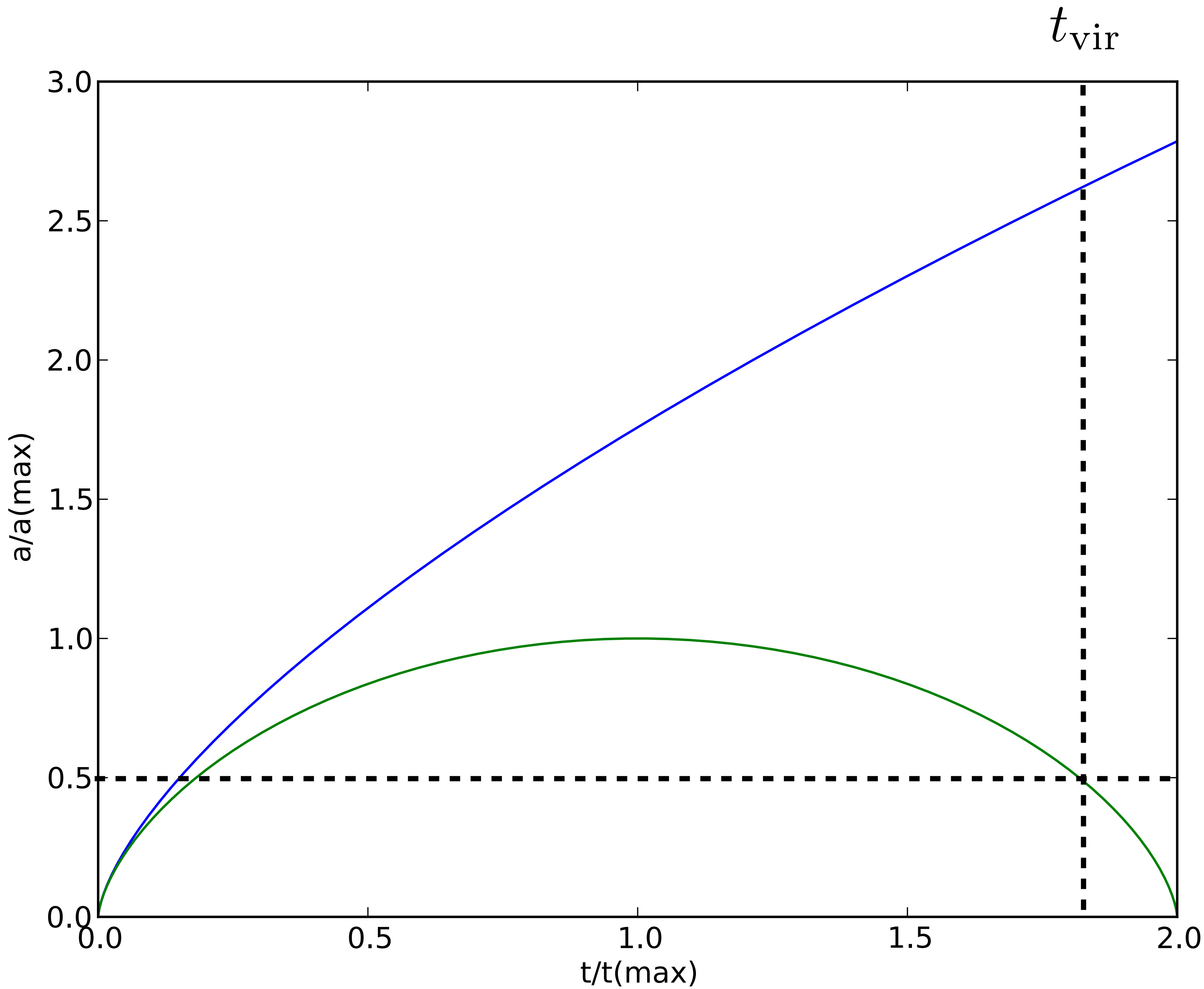


Virial equilibrium:

$$T = -\frac{1}{2}U$$



Virial equilibrium is reached when perturbation has contracted to half its maximum size and 8x the minimum density.



$\rho_{\rm vir}/\rho_0 \approx 150$

Collapse of structure

Key point:

Structures became virialised once their densities reached >100 times the background density

When did structures become virialised?

Individual galaxies:

Today, $\Delta \approx 10^6$

Epoch when $\rho_{gal} \approx 10^2 \langle\rho\rangle$

$$\rho/\rho_0 \approx 10^4 \rightarrow (1+z) \approx 20$$

Galaxy clusters

Today, $\Delta \approx 10^3$

$$\rho/\rho_0 \approx 10 \rightarrow (1+z) \approx 2$$

We expect galaxy clusters to have become virialised relatively recently (some are not yet fully virialised).

The mass function of bound structures

- Investigated by Press & Schechter (1974)
- Basic assumptions:
  - Density spectrum is initially Gaussian
  - Perturbations initially grow linearly, then collapse rapidly when they exceed some threshold amplitude Δ_c.
  - Growth is *hierarchical*: Small perturbations can be part of larger ones

Press-Schechter theory

Suppose the matter consists of randomly distributed particles.

For average volume number density *n(m)* and particle mass *m* the variance on the mass in a unit volume is

$$\sigma^2 = \int_0^\infty m^2 n(m) \mathrm{d}m$$

For fluctuations occupying a volume *V* the variance is

$$\Sigma_V^2 = V\sigma^2$$

Normalized to the mass in the volume:

$$\Sigma_V / M(V) = \frac{\sigma\sqrt{V}}{\rho V} = \frac{\sigma}{\rho\sqrt{V}}$$

Press-Schechter theory

Normalised to the mass in the volume:

$$\Sigma_V/M(V) = \frac{\sigma\sqrt{V}}{\rho V} = \frac{\sigma}{\rho\sqrt{V}}$$

The relative fluctuations in *M* within volume *V*:

$$\Delta \equiv \frac{\Sigma_V}{\langle M(V)\rangle} = \frac{M(V) - \langle M(V)\rangle}{\langle M(V)\rangle}$$

are normally distributed with standard deviation

$$\Delta_\star = \sigma/(\rho\sqrt{V})$$

That is, the probability distribution of over-densities Δ within *V* is

$$p(\Delta, V) = \frac{1}{\sqrt{2\pi}\Delta_\star}\exp\left(-\frac{1}{2}\frac{\Delta^2}{\Delta_\star^2}\right)$$

Press-Schechter theory

Probability distribution of overdensities Δ:

$$p(\Delta, V) = \frac{1}{\sqrt{2\pi}\Delta_\star} \exp\left(-\frac{1}{2}\frac{\Delta^2}{\Delta_\star^2}\right)$$

Next: Probability distribution of *bound* overdensities.

Assume

- Fluctuations with $\Delta > \Delta_{crit}$ at some scale factor a_2 are bound
- Growth is linear: $\Delta(a_2) = (a_2/a_1)\ \Delta(a_1)$ for scale factors a_1 and a_2.

Fluctuations that are critical at a_2 then had an "initial" density contrast

$$\Delta_1 = \frac{a_1}{a_2}\Delta_{\rm crit}$$

Press-Schechter theory

Probability that a volume contains a bound fluctuation by a_2 is then

$$P_{\rm bound} = \int_{\Delta=\Delta_1}^{\infty} p(\Delta, V)\, {\rm d}\Delta$$

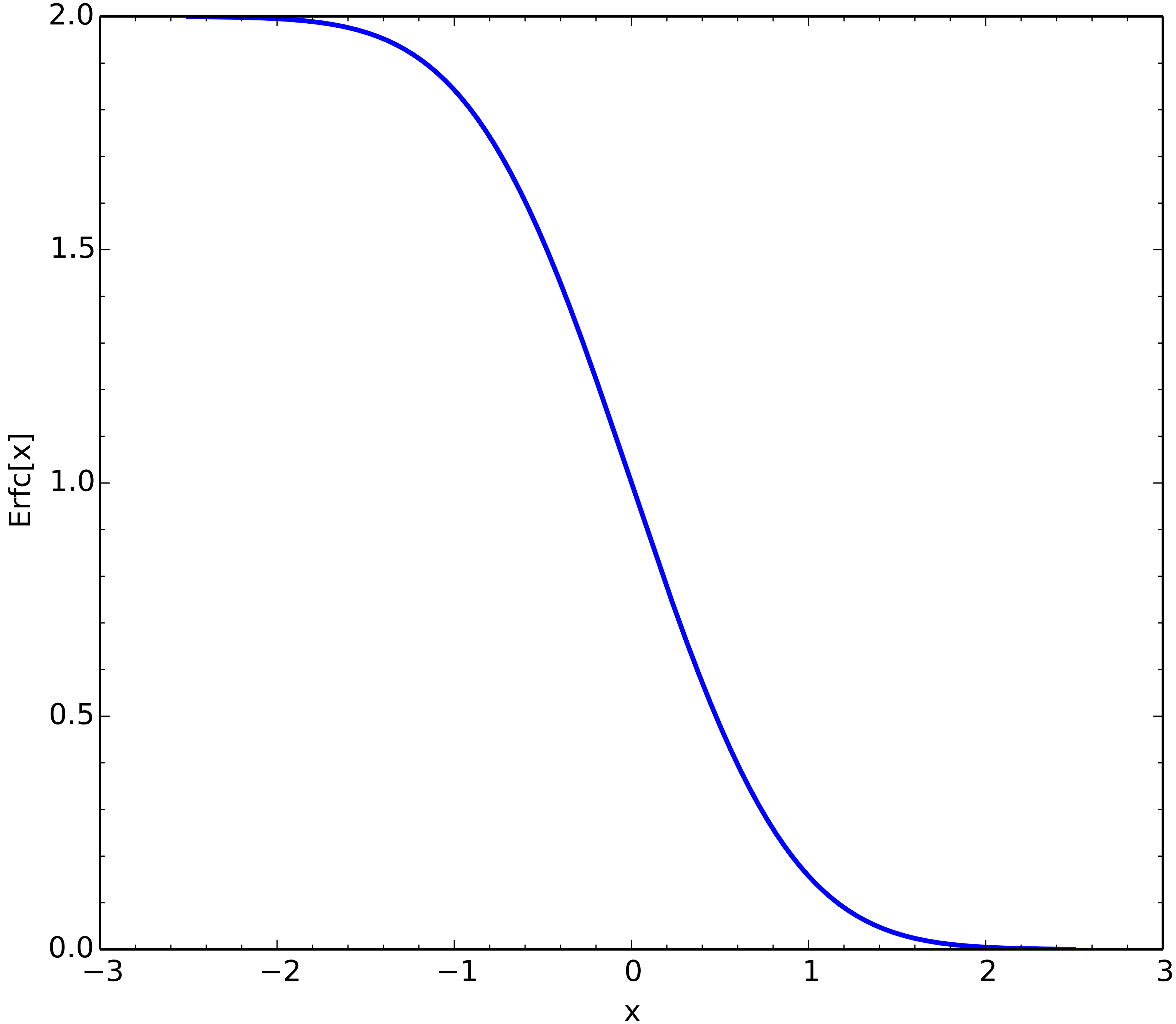
$$= \frac{1}{\sqrt{2\pi}\Delta_\star} \int_{\Delta=\Delta_1}^{\infty} \exp\left(-\frac{1}{2}\frac{\Delta^2}{\Delta_\star^2}\right) {\rm d}\Delta$$

$$= \frac{1}{2}{\rm erfc}\left(\frac{\Delta_{\rm crit} a_1}{\sqrt{2}\Delta_\star a_2}\right)$$

Inserting $\Delta_\star = \sigma/(\rho_1\sqrt{V})$ we then have

$$P_{\rm bound}(V) = \frac{1}{2}{\rm erfc}\left(\frac{\Delta_{\rm crit}\rho_1\sqrt{V} a_1}{\sqrt{2}\sigma a_2}\right)$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2)\,\mathrm{d}t$$



Press-Schechter theory

$$P_{\rm bound}(V) = \frac{1}{2}{\rm erfc}\left(\frac{\Delta_{\rm crit}\rho_1\sqrt{V}a_1}{\sqrt{2}\sigma a_2}\right)$$

Since fluctuations are small initially, $M \approx \rho_1 V$ so

$$P_{\rm bound}(M) = \frac{1}{2}{\rm erfc}\left(\frac{\Delta_{\rm crit}\sqrt{M\rho_1}a_1}{\sqrt{2}\sigma a_2}\right)$$

This is the fraction of fluctuations with mass *M* that have collapsed by a_2.

Some of these will be part of larger collapsed volumes. The fraction of *independent* collapsed fluctuations is therefore

$$\frac{{\rm d}P_{\rm bound}}{{\rm d}M} = \frac{\rm d}{{\rm d}M}\frac{1}{2}{\rm erfc}\left(\frac{\Delta_{\rm crit}\sqrt{M\rho_1}}{\sqrt{2}\sigma}\frac{a_1}{a_2}\right)$$

Press-Schechter theory

The fraction of *independent* collapsed fluctuations is

$$\frac{\mathrm{d}P_{\mathrm{bound}}}{\mathrm{d}M} = \frac{\mathrm{d}}{\mathrm{d}M}\frac{1}{2}\mathrm{erfc}\left(\frac{\Delta_{\mathrm{crit}}\sqrt{M\rho_1}}{\sqrt{2}\sigma}\frac{a_1}{a_2}\right)$$

$$= -\frac{2^{-3/2}}{\sqrt{\pi}}\frac{\Delta_{\mathrm{crit}}\sqrt{\rho_1}}{\sigma}\frac{a_1}{a_2}M^{-1/2}\exp\left[-\frac{1}{2}\frac{\Delta^2_{\mathrm{crit}}\rho_1}{\sigma^2}\left(\frac{a_1}{a_2}\right)^2 M\right]$$

Since fluctuations of mass *M* (initially) occupied a volume $V=M/\rho_1$, the *number* density is

$$\frac{\mathrm{d}N}{\mathrm{d}M} = \rho_1 M^{-1}\frac{\mathrm{d}P}{\mathrm{d}M}$$

which is of the form

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-3/2}\exp(-M/M^\star)$$

with $M^\star \propto a^2$

Press-Schechter mass function:

$$\frac{\mathrm{d}N}{\mathrm{d}M} \propto M^{-3/2} \exp(-M/M^{\star})$$

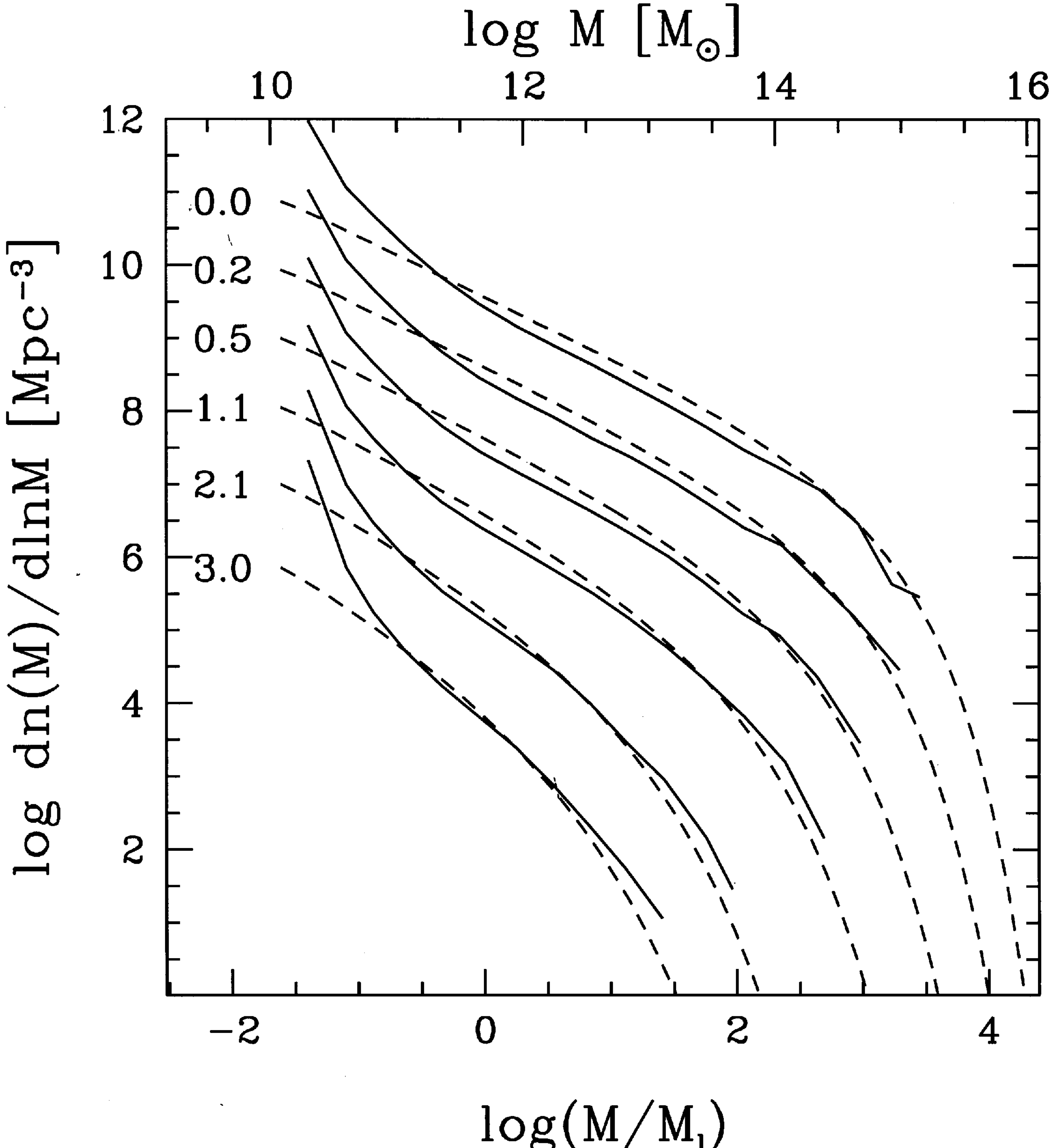
Low masses (M<<M*):
Power-law shape

High masses (M>M*):
Exponential cut-off

M* scales with a^2.

Binggeli (1987)

Comparison with simulations



Solid curves:
Numerical simulation

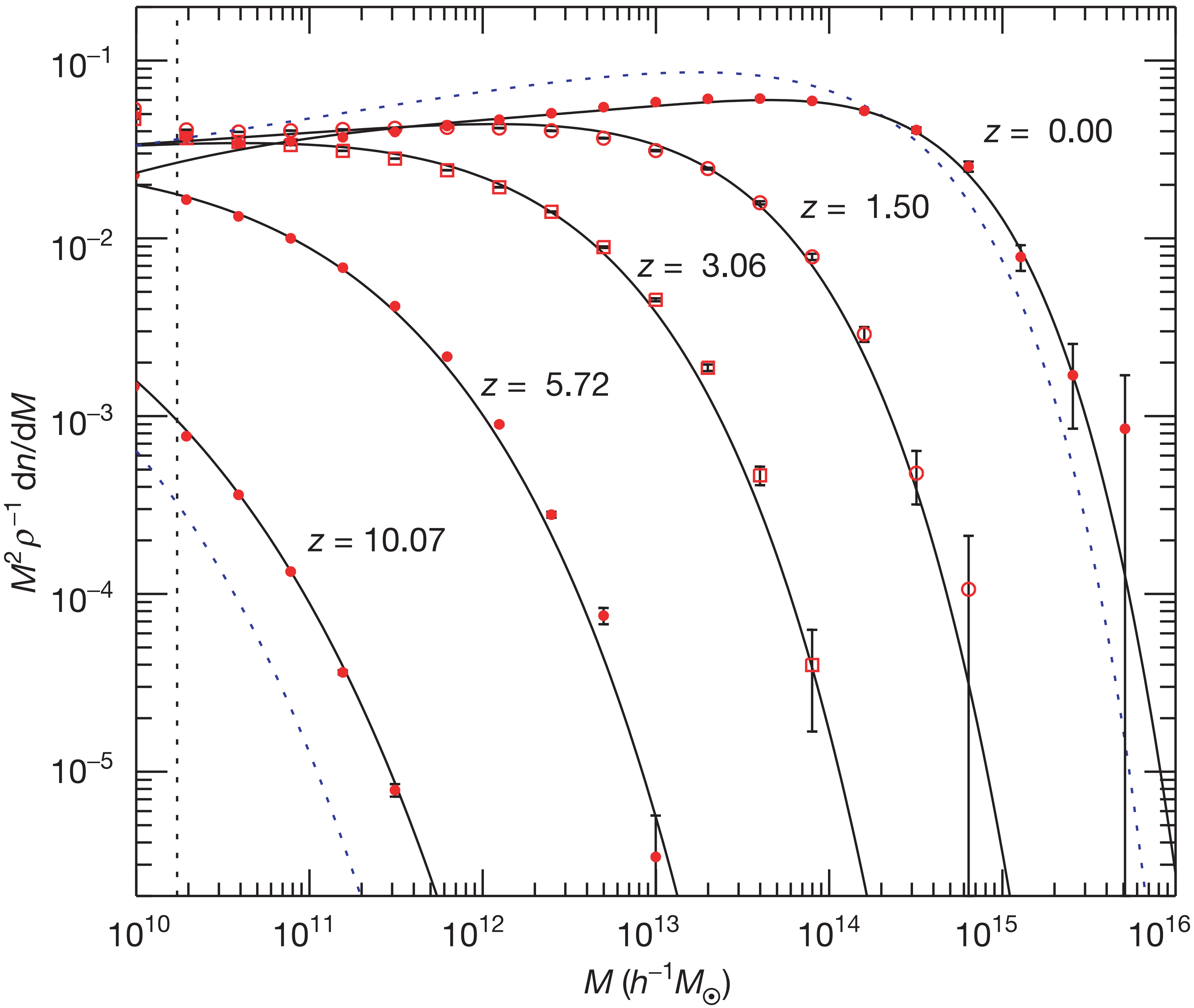
Dashed curves:
Press-Schechter model

Strong evolution in Schechter mass expected!

Generic feature of hierarchical structure formation models.

Somerville et al. (2000)

Comparison with simulations



Solid curves:
Millennium simulation

Dotted curves:
Press-Schechter model

Massive galaxy clusters should be rare at high redshifts.

Springel et al. (2005)

Schechter function for *galaxies*

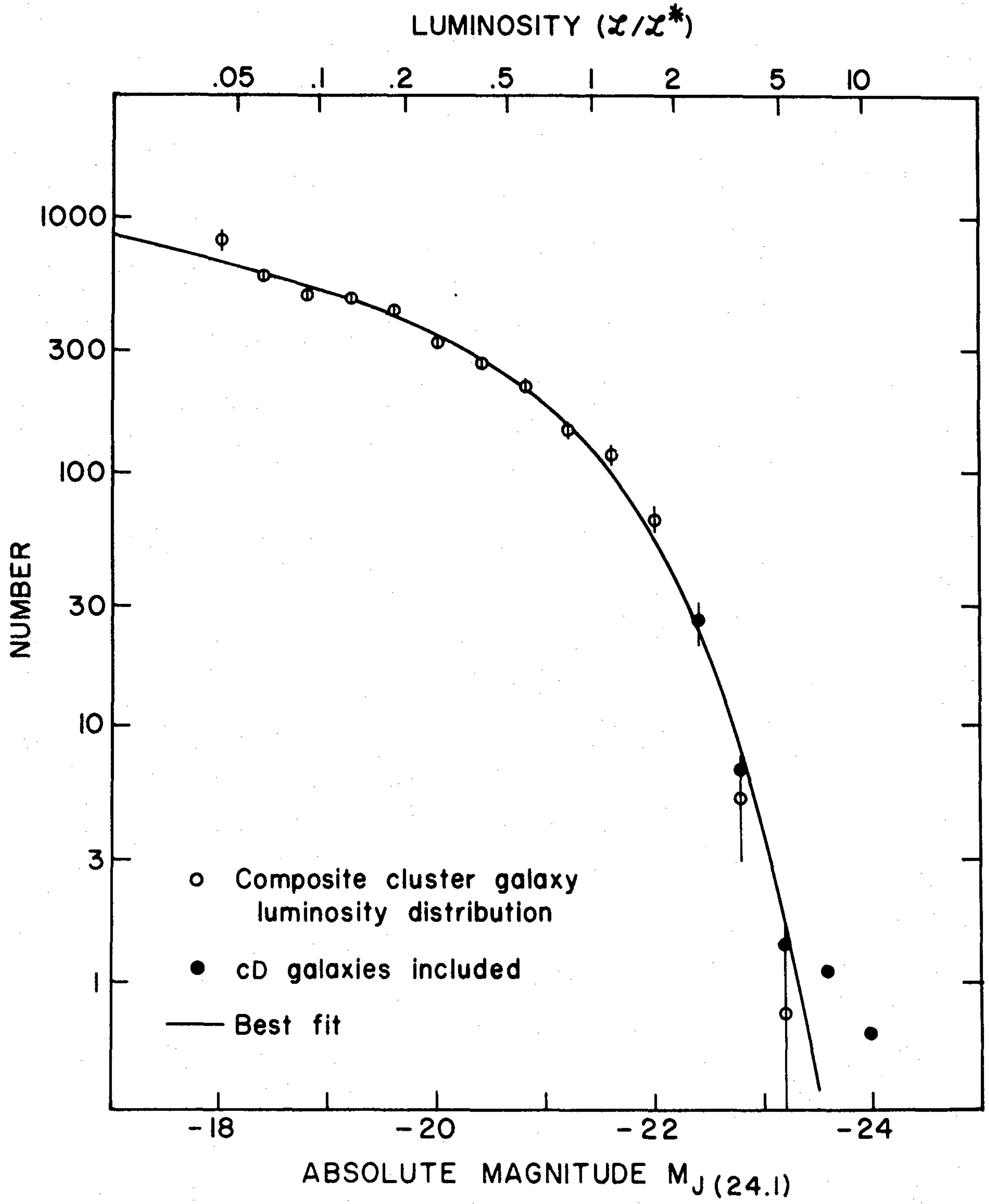


FIG. 2.—Best fit of analytic expression to observed composite cluster galaxy luminosity distribution. Filled circles show the effect of including cD galaxies in composite.

Assume relation of form

$$\frac{\mathrm{d}N}{\mathrm{d}L} \propto \left(\frac{L}{L^\star}\right)^\alpha \exp\left(-\frac{L}{L^\star}\right)$$

Inspired by the Press-Schechter analysis.

General case: Best fit for
α = -1.24 +/- 0.19
M_B* = -20.60 +/- 0.11

Cluster galaxies: Best fit for
α = -1.24 +/- 0.05
M_B* = -21.41 +/- 0.10

But cD galaxies must be excluded!

Schechter (1976)

In one of the opening volleys of the science wars, Abraham Maslow wrote [13] wrote:

> I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.

It's true. My hammer was the power law with an exponential cutoff and I hammered on the luminosity function for galaxies [19].

Schechter (2002)

Problems with Press-Schechter analysis

- Assuming spherical symmetry
- Schechter mass predicted at z=0 is of order 10^{15} $M_\odot$ - applicable to galaxy *clusters*.
- The MF for *individual galaxies* also follows a Schechter function, but with much lower M*
- Why the two different characteristic masses?
- Important physics missing: *baryons* (dissipation and feedback).

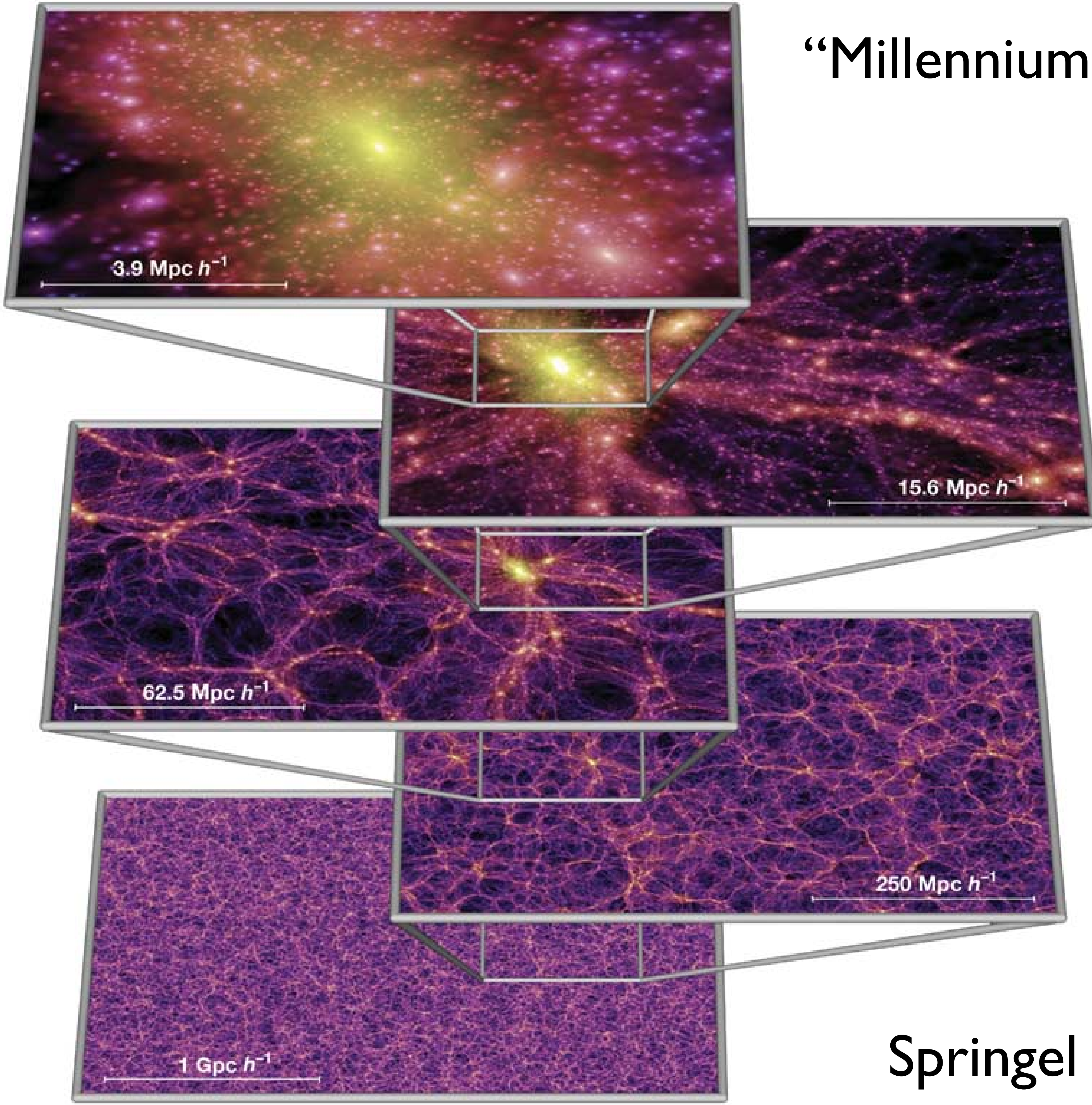
Going further

- Major simplification in the top-hat model and Press-Schechter formalism: spherical symmetry
- Better: treat perturbations as tri-axial ellipses
- Tri-axial perturbations collapse across the *shortest* axis first and form *pancakes* and *filaments* (Zeldovich 1970)
- Most general approach: follow evolution with numerical N-body simulations

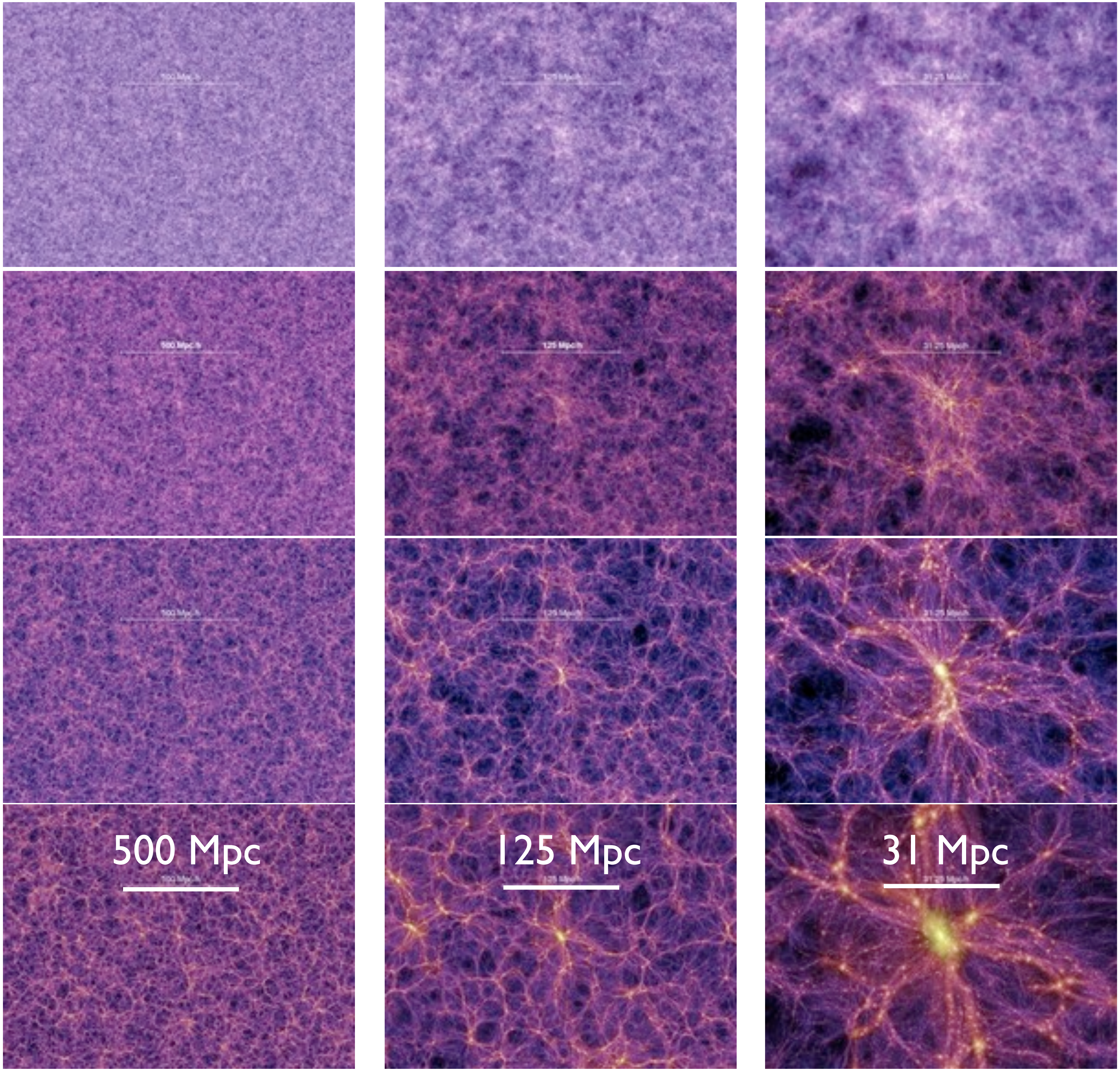
Example N-body sim.

- "Millennium" simulation, Springel et al. (2005):
- Simulation started at z=127 with a random realization of fluctuation power-spectrum from CMBFAST code
- Cosmological parameters:
Ω_0 = 0.25, Ω_B=0.045, Ω_Λ=0.75, H_0=73 km/s/Mpc
- Simulation volume $(685 \text{ Mpc})^3$, 10^{10} particles (each individual particle has $M \sim 10^9\ M_\odot$)

“Millennium” simulation



Springel et al. (2005)

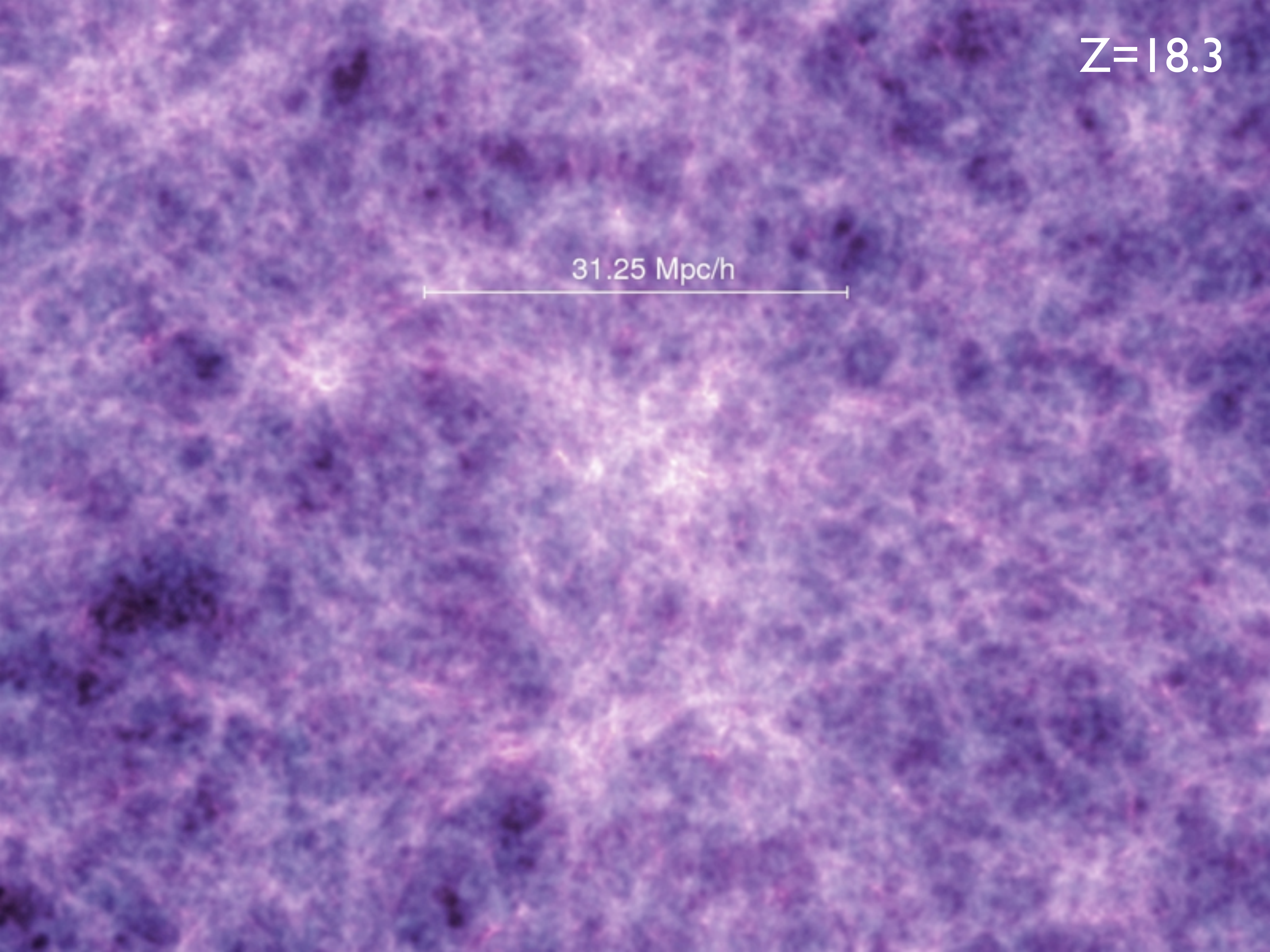


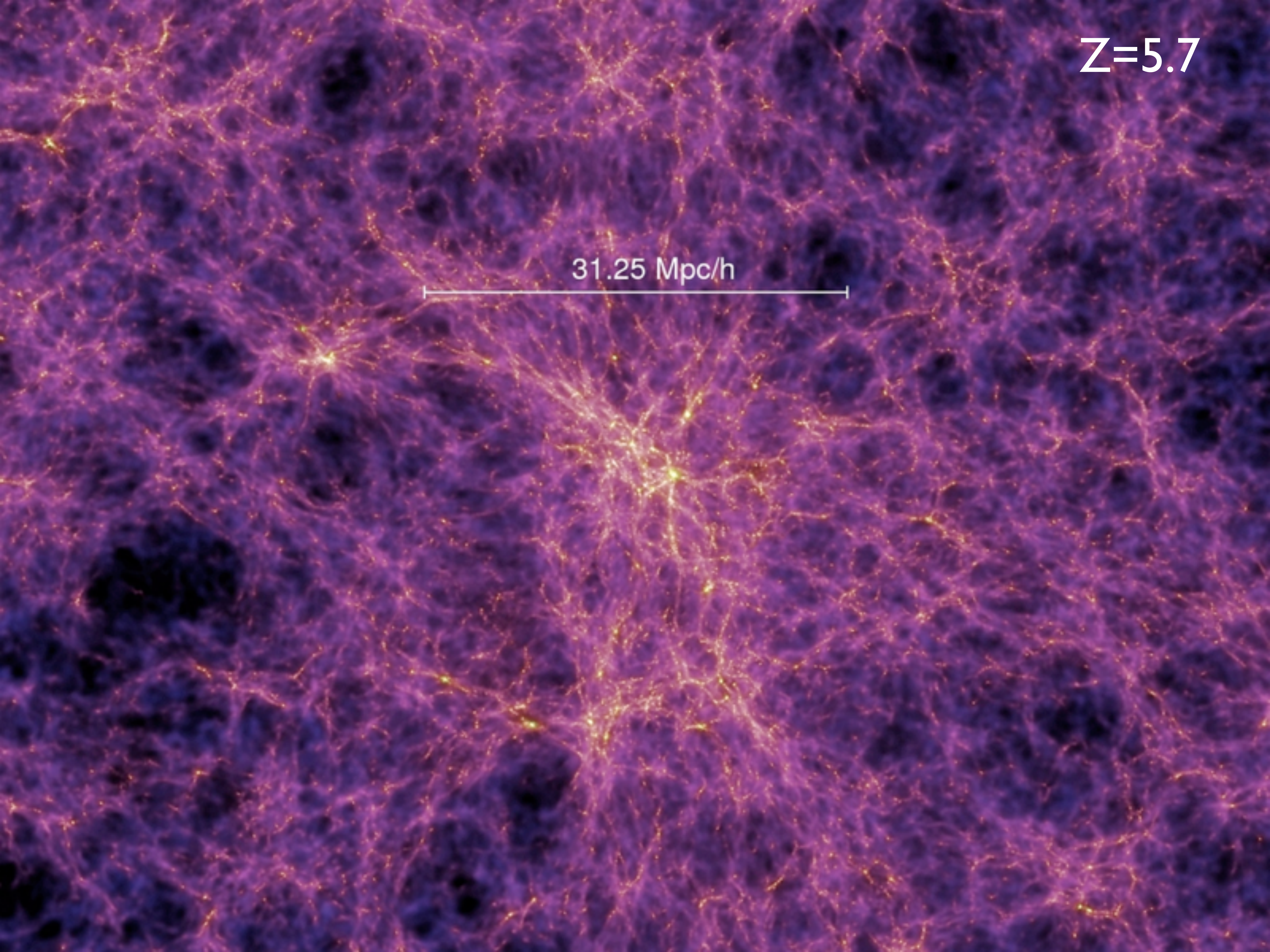
z=18.3 (0.21 Gyr)

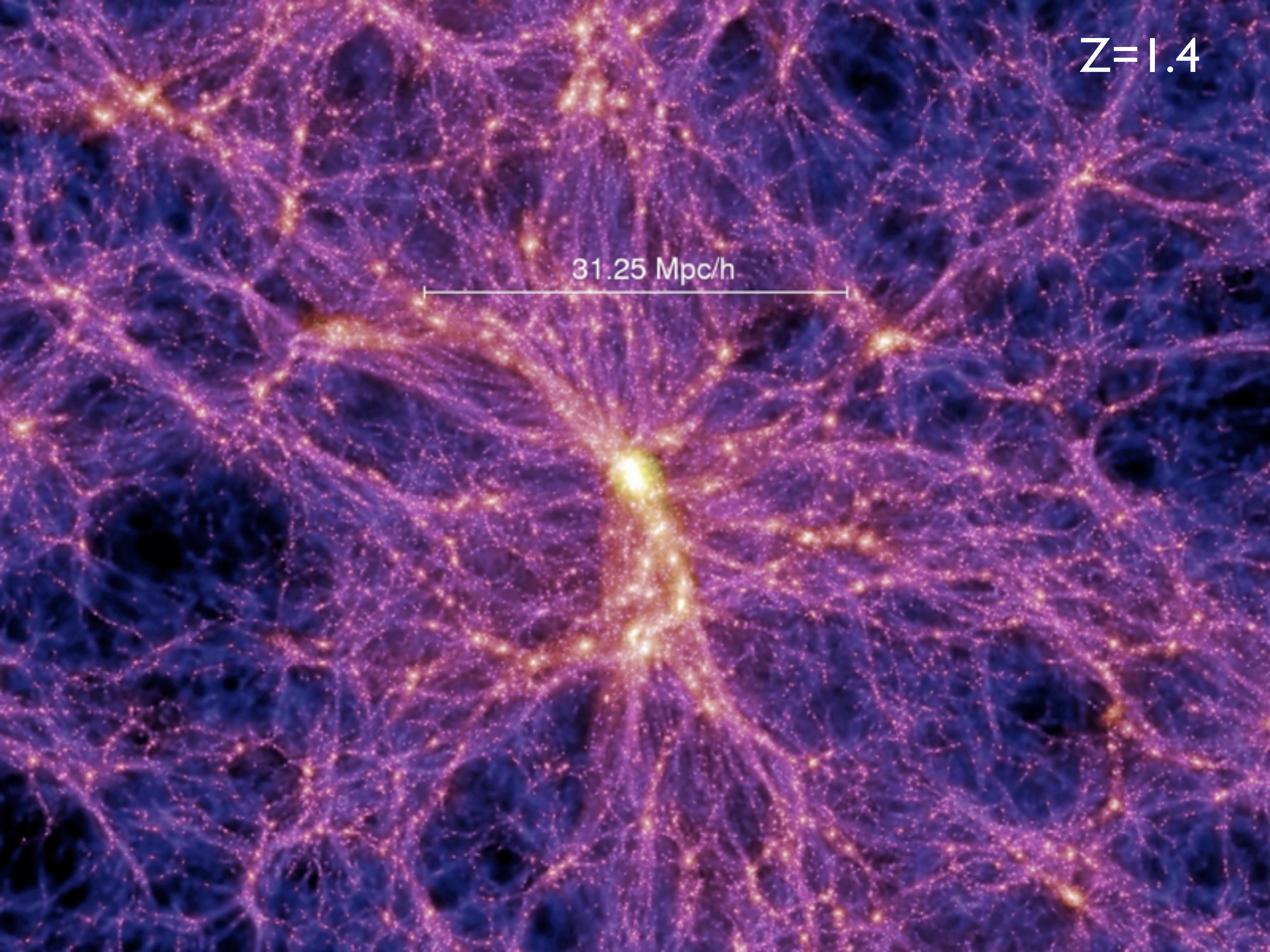
z=5.7 (1.0 Gyr)

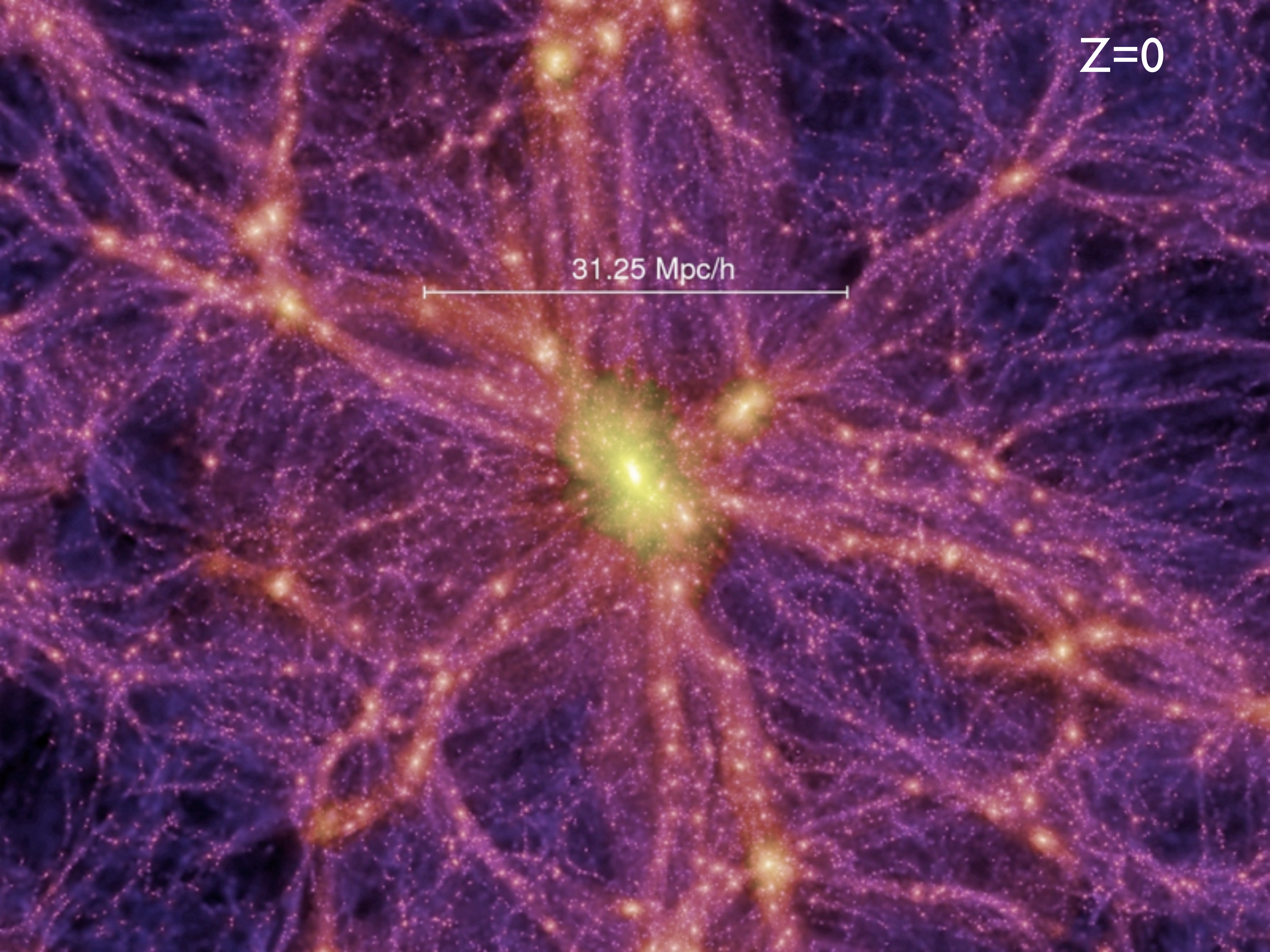
z=1.4 (4.7 Gyr)

z=0 (13.6 Gyr)


Z=18.3
31.25 Mpc/h


Z=5.7
31.25 Mpc/h


Z=1.4
31.25 Mpc/h


Z=0
31.25 Mpc/h