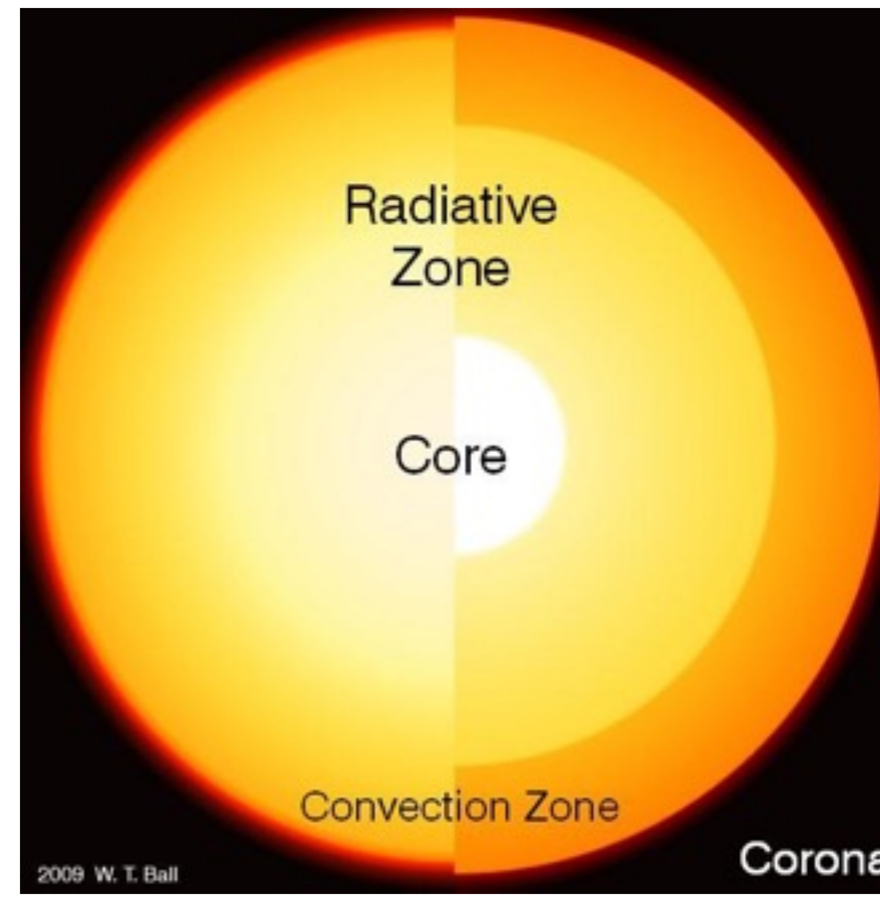
Stellar atmospheres and spectra

What can we learn?

- In most stars, for most of their lifetime, the products of nucleosynthesis remain confined to the stellar interior

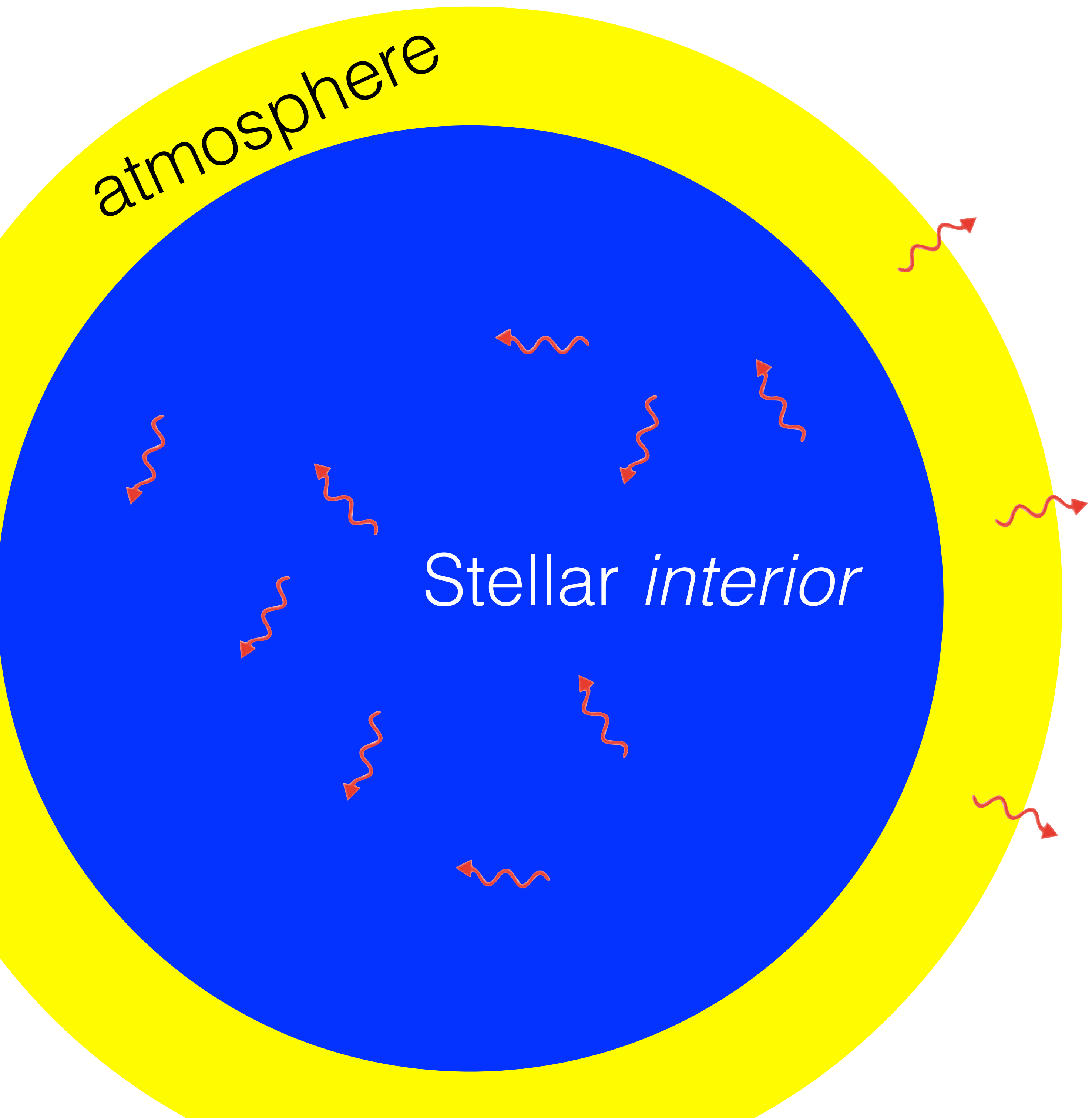
- The composition of the stellar atmosphere remains (mostly) unchanged for most of a star's lifetime

- Stellar atmospheres provide a "fossil record" of the history of (chemical) evolution in a galaxy



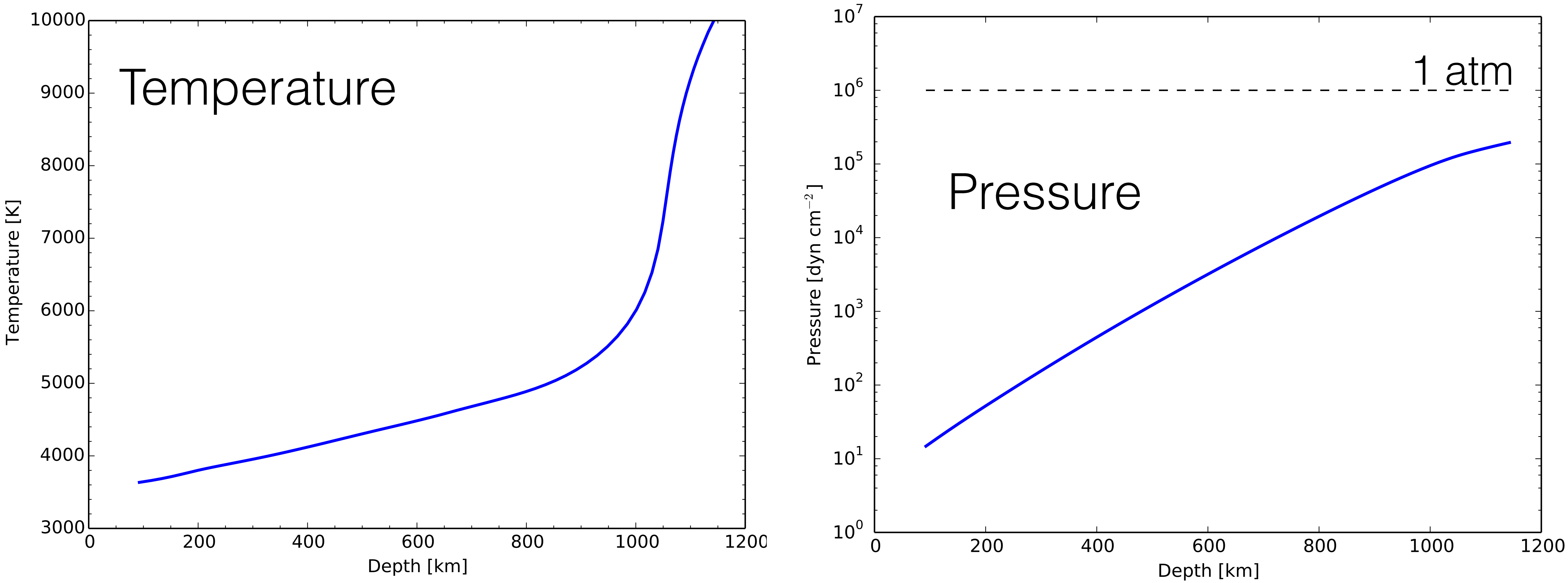
Internal structure of the Sun

Stellar atmospheres



- Stellar *interior*:
  High densities - photons get scattered many times, cannot escape
- Stellar *atmosphere*:
  Outer “layer” where photons can escape (and hence be observed by us)
- Usually, the atmosphere is very thin compared to the radius of a star (about 500 km for the Sun).

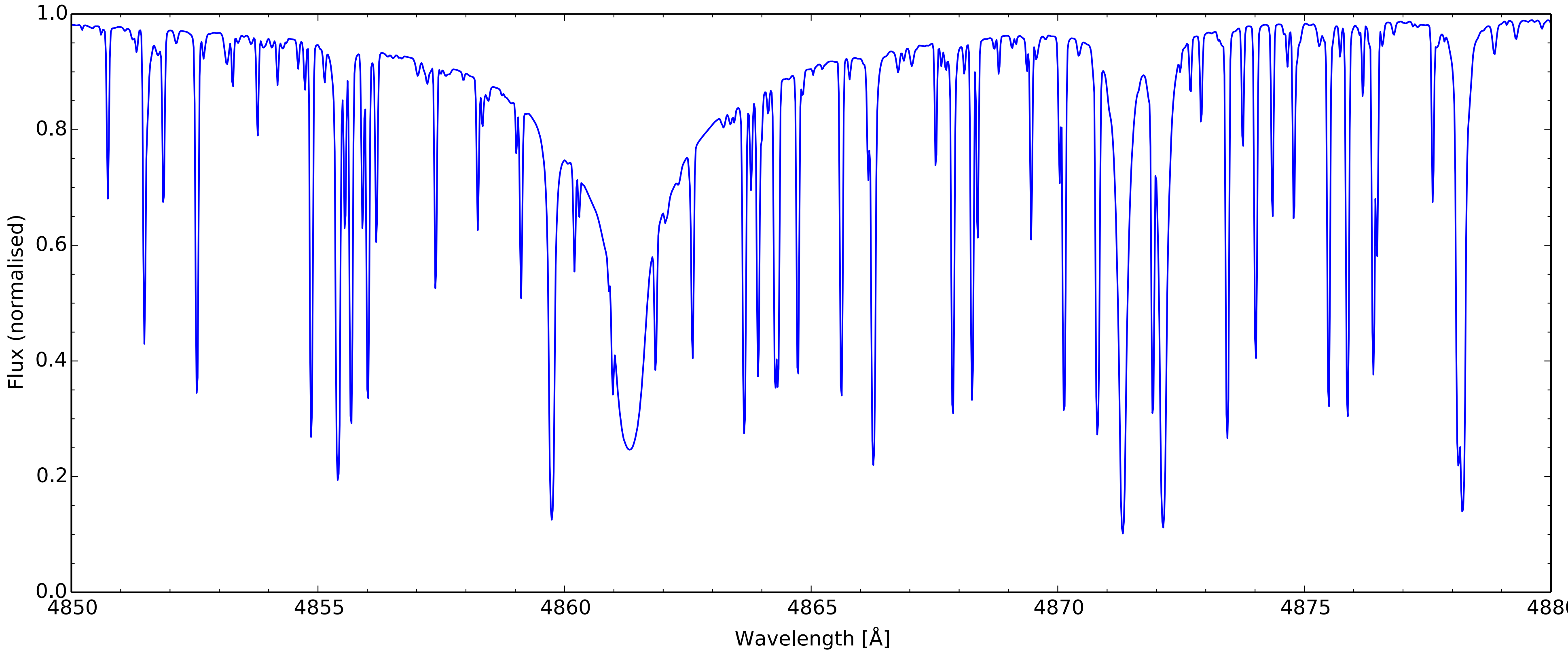
Structure of the Solar atmosphere



Stellar model atmosphere:

- Temperature (*T*) and pressure (*P*) as a function of depth.
- We need to know these physical properties before we can calculate the emergent spectrum.

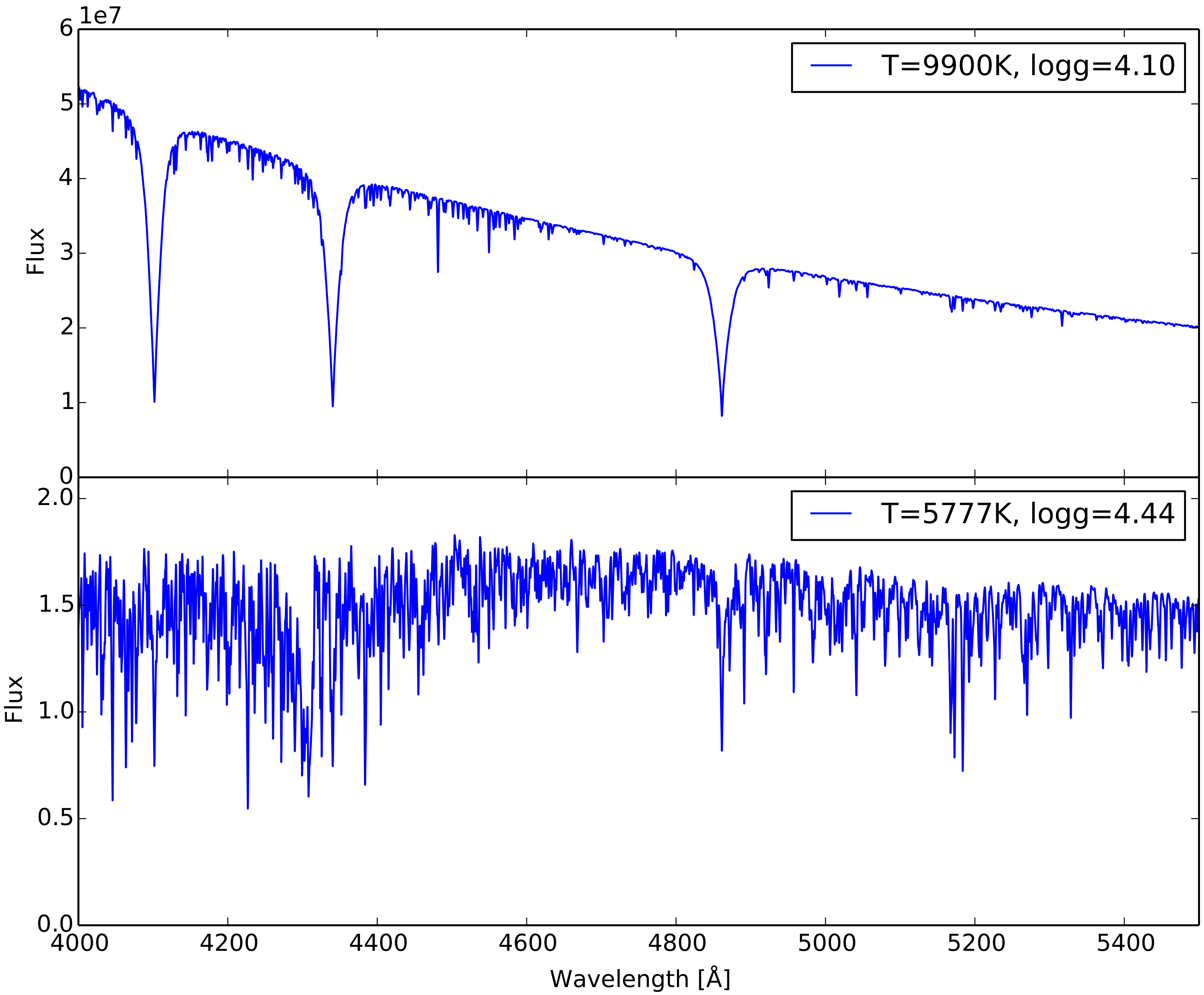
Model spectrum for the Sun



Each line corresponds to a transition between two energy levels in a specific atom/ion.
Line strengths depend on:

- Abundance of element - *chemical composition*.
- Fraction of atoms in relevant state of ionization/excitation - *statistical physics*.
- Transition probabilities ("oscillator strengths") - *atomic physics*.
- Absorption/emission along line-of-sight - *radiative transfer*.

Same composition, different stars



“Vega-like” star

- Strong hydrogen lines
- Weak metal lines

“Sun-like” star

- Weak hydrogen lines
- Strong metal lines

Local Thermodynamic Eq.

Stellar atmosphere+spectra calculations are often carried out under the assumption of *Local Thermodynamic Equilibrium* (LTE).

In LTE, the excitation and ionization of atoms do not depend on the details of the radiation field, but only on the *temperature*.

Excitation: *Boltzmann* equation
Ionization: *Saha's* equation

The Boltzmann equation

Boltzmann equation:

$$\frac{N_j}{N_i} = \frac{g_j}{g_i} e^{-E_{ij}/kT} = \frac{g_j}{g_i} e^{-h\nu/kT}$$

$kT >> E_{ij}$ means more atoms in the *upper* state

$kT << E_{ij}$ means more atoms in the *lower* state

For hydrogen:

$g_1=2$, $g_2=8$, $E_{12} = 1.6\times10^{-18}$ J

$T = 5777$ K → $kT = 8\times10^{-20}$ J → $N_2/N_1 = 8\times10^{-9}$

$T = 10000$ K → $kT = 1.4\times10^{-19}$ J → $N_2/N_1 = 3.7\times10^{-5}$

In Vega: N_2/N_1 about 4000 times higher than in the Sun

The Saha equation

Saha's equation:

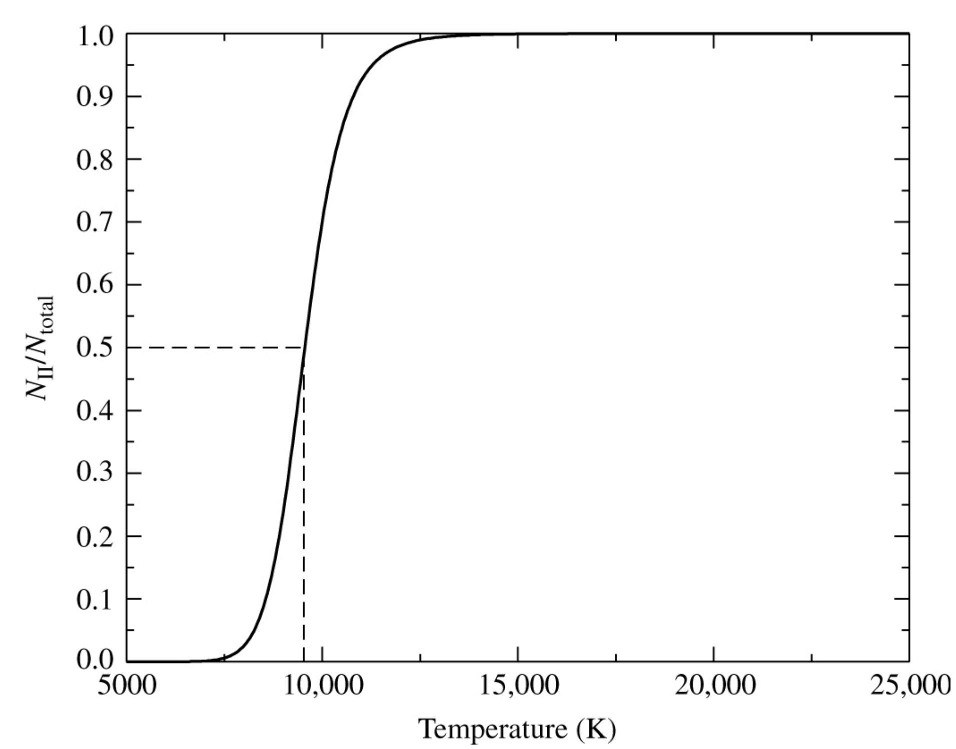
$$\frac{N_{\text{upper}} N_e}{N_{\text{lower}}} = 2 \left(\frac{2\pi m_e k T}{h^3} \right)^{3/2} \frac{U_{\text{upper}}}{U_{\text{lower}}} e^{-I/kT}$$

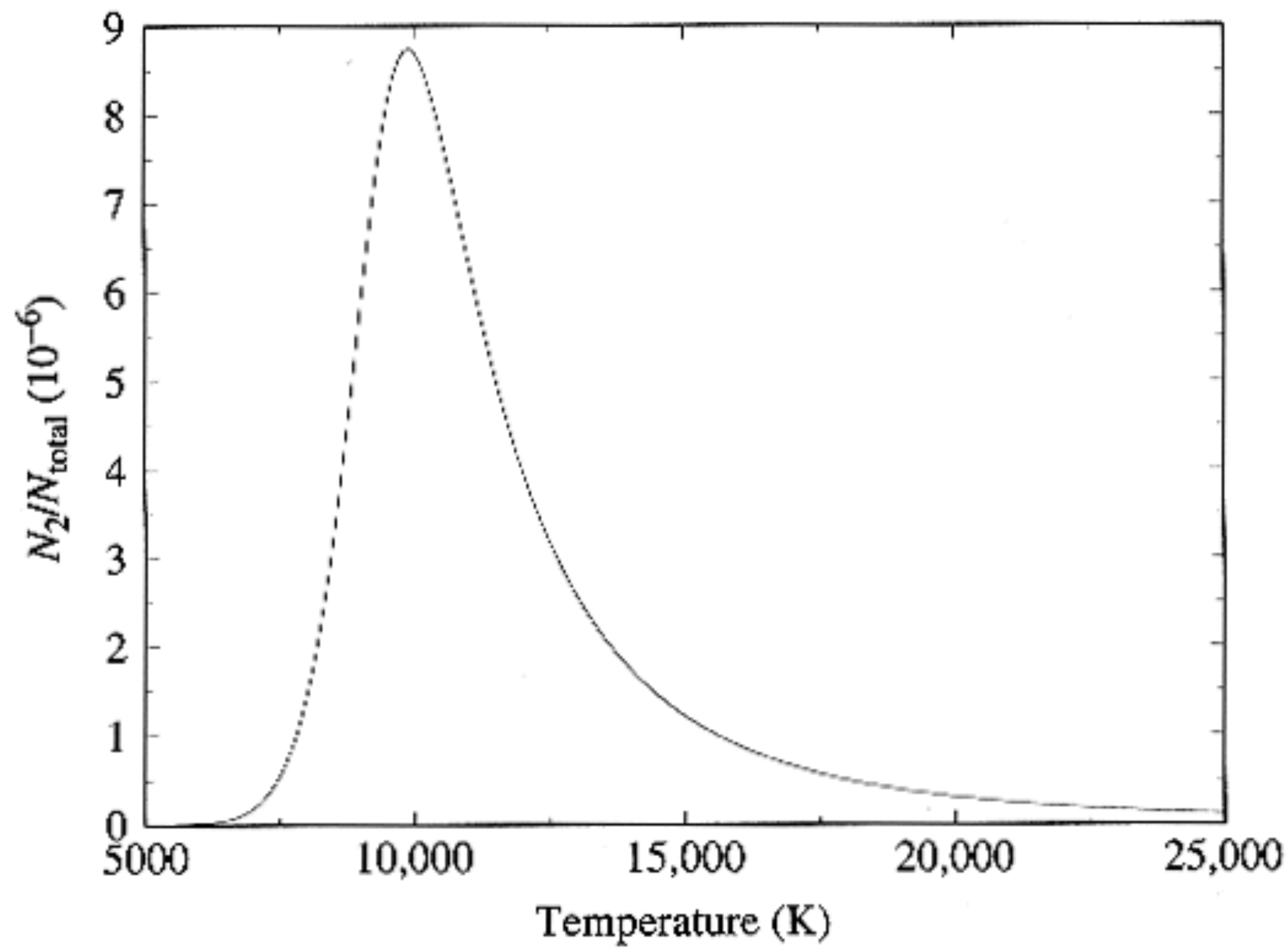
for *partition functions* $U = \sum g_i e^{-E_i/kT}$

Gives the ratio of atoms in two ionization stages (lower, upper).

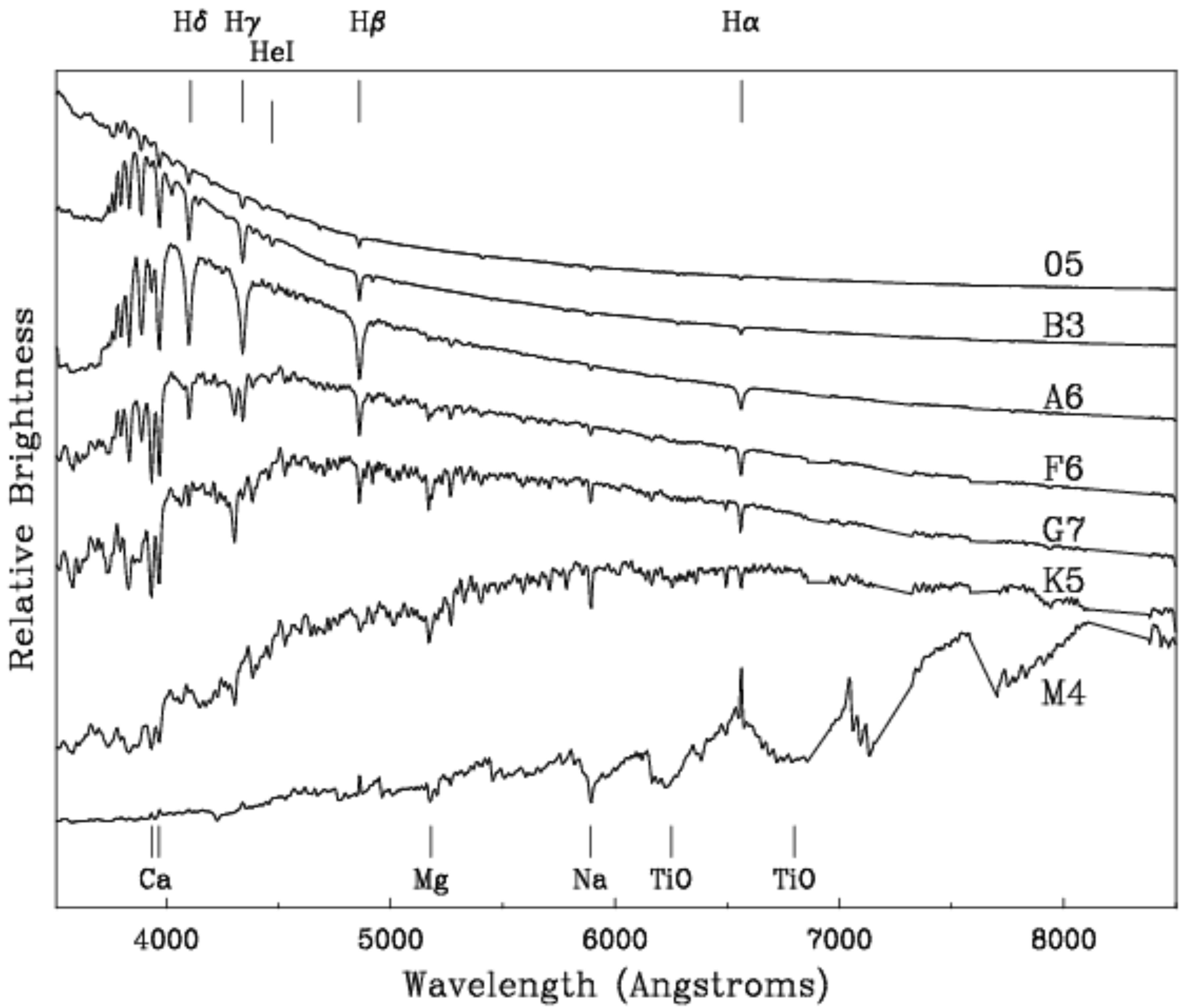
- *I* = ionization energy
- N_e = electron density

For T>10000 K, most H atoms are ionized!



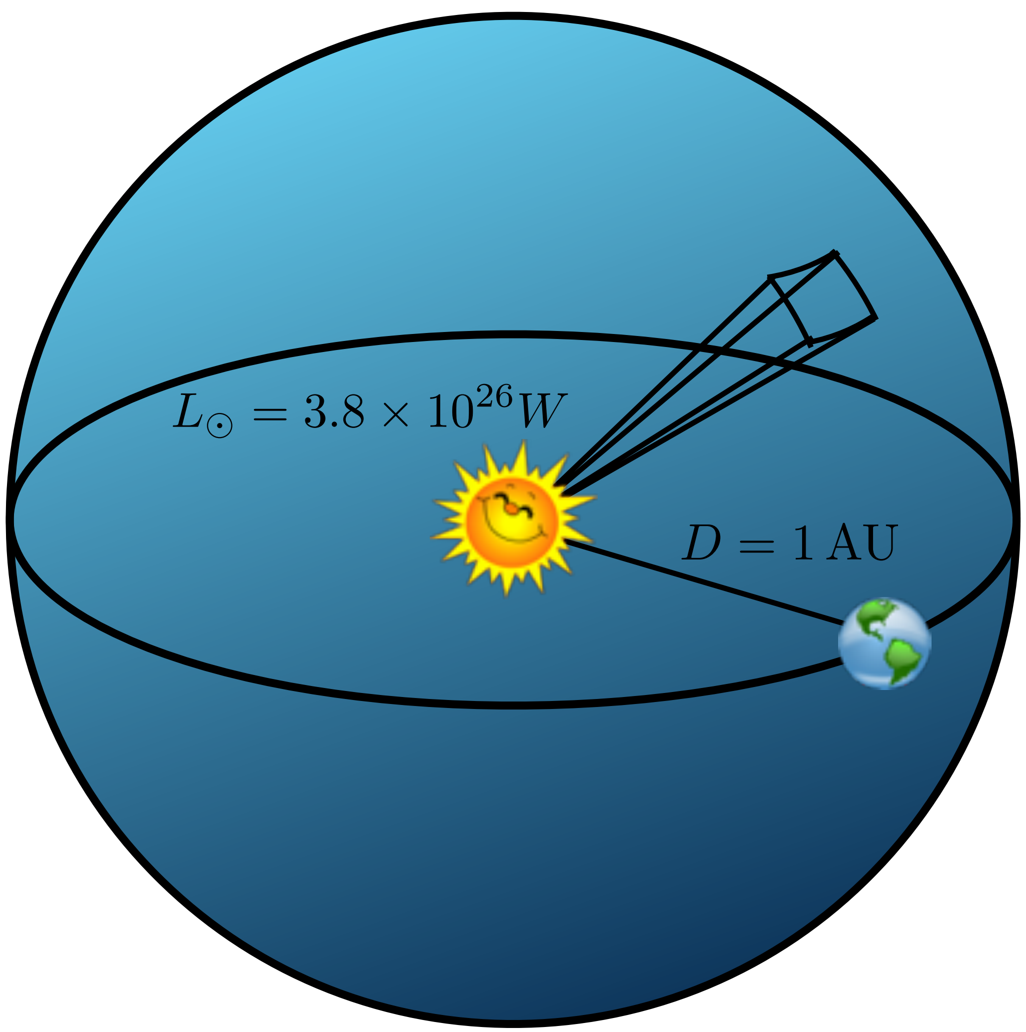


Fraction of hydrogen atoms in n=2 level, relative to total.
(Carroll and Ostlie, *Modern Astrophysics*)


Hδ
Hγ
HeI
Hβ
Hα
O5
B3
A6
F6
G7
K5
M4
Ca
Mg
Na
TiO
TiO
4000
5000
6000
7000
8000
Wavelength (Angstroms)
Relative Brightness

Basic concepts - Flux

Flux: Energy passing through a surface of *unit area* per *unit time*.
Units: W m^{-2}



Example:

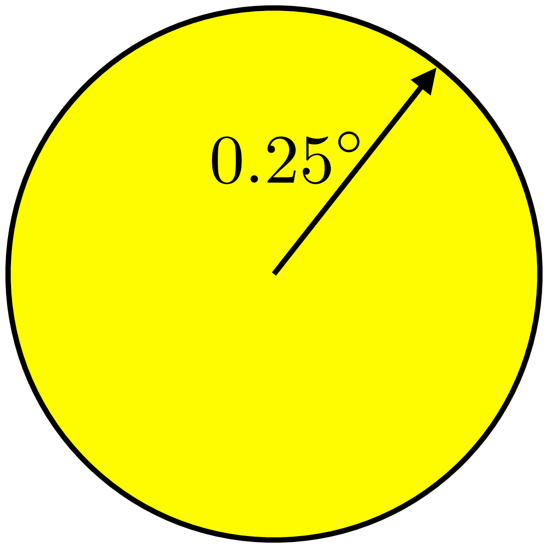
Luminosity of Sun: $L_\odot = 3.8 \times 10^{26}$ W

Flux measured at Earth:

$$F = L_\odot / (4\,\pi\,D^2)$$
$$= 1350 \text{ W m}^{-2}$$

Basic concepts - Intensity

Intensity: Flux per *solid angle*. Units: W m^{-2} sr^{-1}.



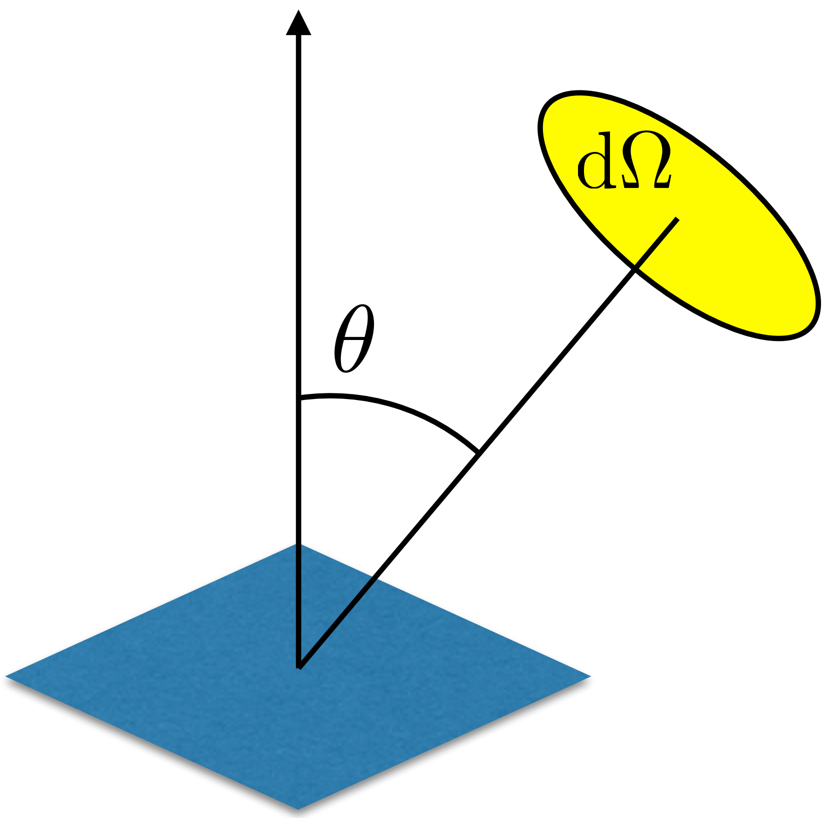
Example:

Flux of Sunlight: F = 1350 W m^{-2}

Solid angle, $\Omega = \pi\,(0.25^{\circ})^2 = 0.20\ \mathrm{deg}^2$.
Intensity = F/Ω = 6750 W m^{-2} deg^{-2}

Or, for Ω in sr: $\Omega = \pi\,(0.25^{\circ} \times \pi/180)^2$ sr
Intensity = F/Ω = 2.3×10^{7} W m^{-2} sr^{-1}

Flux and Intensity cont'd



Flux of radiation with intensity I from solid angle dΩ at angle θ with respect to normal of surface:

$$\mathrm{d}F = I\cos\theta \mathrm{d}\Omega$$

Flux from all directions:

$$F = \int I\cos\theta\, \mathrm{d}\Omega$$

Mean intensity:

$$J = \frac{1}{4\pi}\int I \mathrm{d}\Omega$$

Basic concepts - Effective Temperature

For a black body, we have $$F_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1}$$

Integrating over all frequencies, we have the Stephan-Boltzmann law:

$$F = \frac{2\pi^5 k^4}{15h^3c^2}T^4 = \sigma T^4$$

For a spherical body with radius *R*, the luminosity is then

$$L = 4\pi R^2\sigma T^4$$

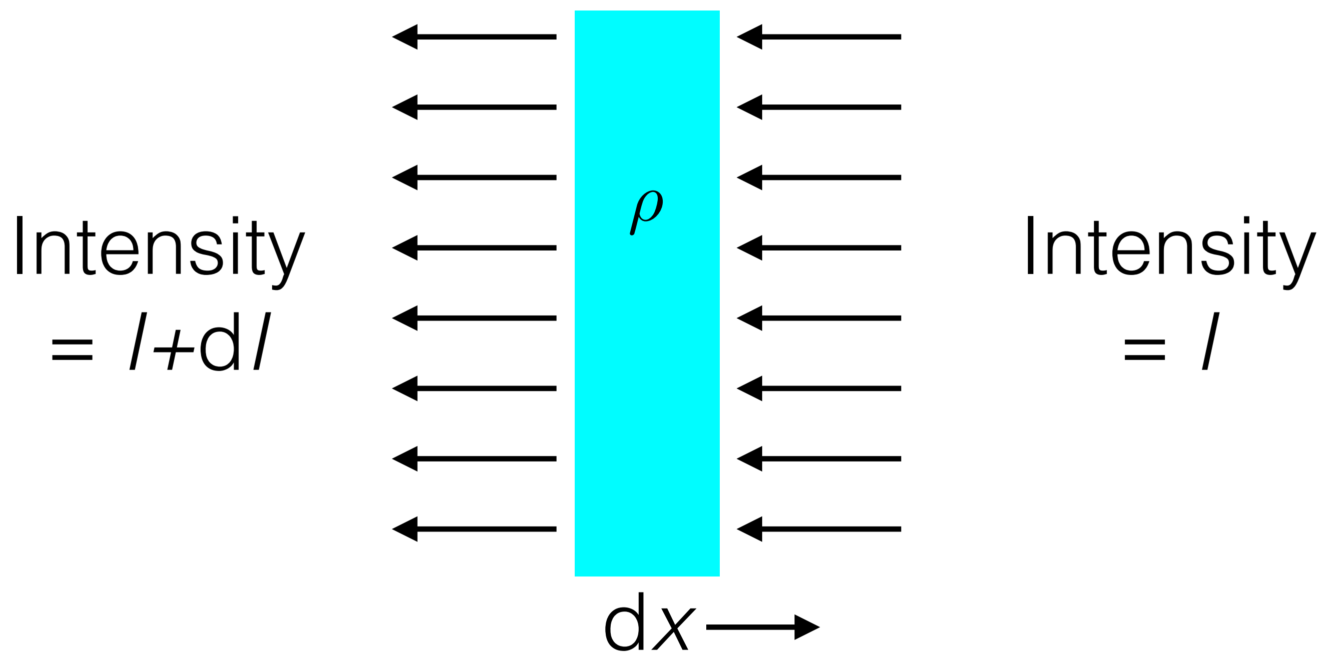
For a star of luminosity *L* and radius *R*, we *define* the effective temperature $T_{\rm eff}$ as

$$T_{\rm eff} \equiv \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4}$$

even though stellar spectra are not, in general, black bodies

Radiative transfer

Suppose a radiation field is propagating in a medium of density ρ:



Amount of radiation absorbed: $dI_{abs} = \kappa \rho I dx$

κ = *absorption* coefficient

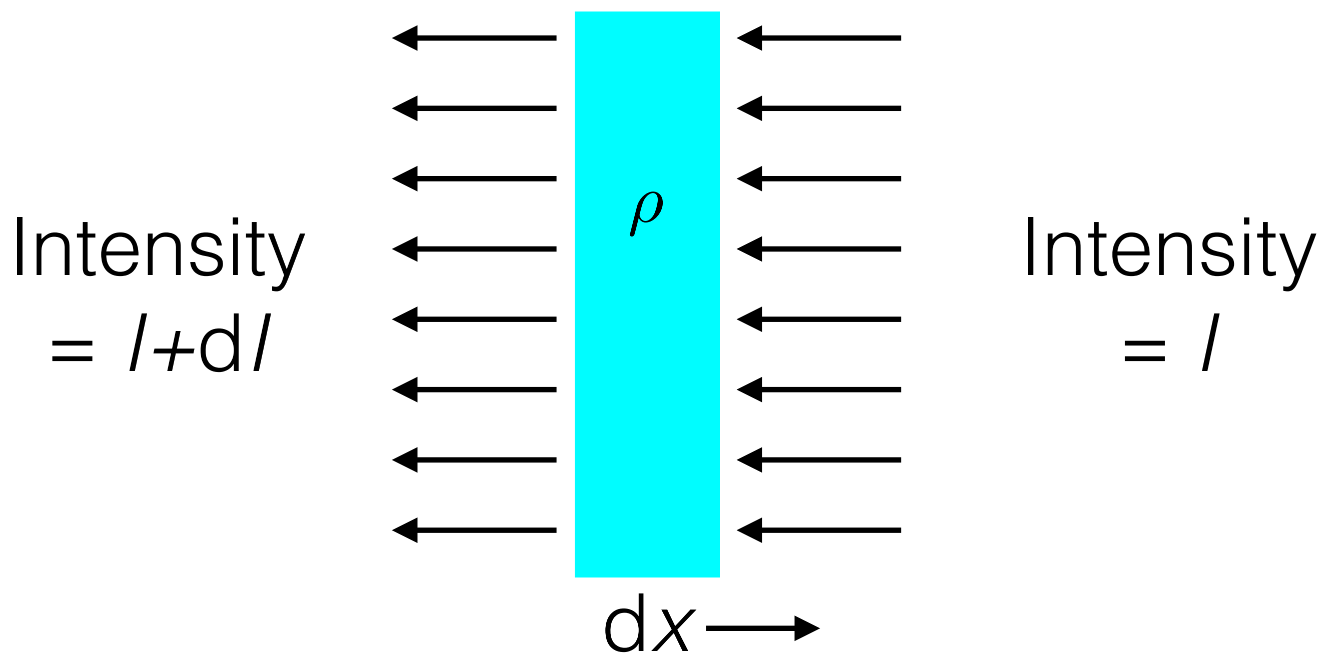
Amount of radiation emitted: $dI_{em} = j \rho dx$

j= *emission* coefficient

Total intensity change: $dI = \kappa \rho I dx - j \rho dx$

Note: by convention, *x* increases inwards

Radiative transfer



Total intensity change: dI = κ ρ I dx - j ρ dx

Define *optical depth*: dτ = κρdx

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - \frac{j}{\kappa}$$

Introducing the **source function**, $S = j/\kappa$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S$$

Equation of radiative transfer

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S$$

for source function

$$S \equiv \frac{j}{\kappa}$$

Equation of radiative transfer

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = I - S$$

A formal solution, $I(\tau)$, can be obtained by multiplying by $e^{-\tau}$:

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} e^{-\tau} - I e^{-\tau} = -S e^{-\tau}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} I e^{-\tau} = -S e^{-\tau}$$

Then integrate over τ:

$$I(\tau_a) e^{-\tau_a} - I(\tau_b) e^{-\tau_b} = \int_{\tau_a}^{\tau_b} S e^{-\tau'} \, \mathrm{d}\tau'$$

$$I(\tau_a) = I(\tau_b) e^{\tau_a - \tau_b} + \int_{\tau_a}^{\tau_b} S e^{\tau_a - \tau'} \, \mathrm{d}\tau'$$

Equation of radiative transfer

General solution:

$$I(\tau_a) = I(\tau_b)e^{\tau_a - \tau_b} + \int_{\tau_a}^{\tau_b} S e^{\tau_a - \tau'}\,\mathrm{d}\tau'$$

When calculating a model spectrum, the integral generally runs from τ_a=0 (at the surface) to $\tau_b \gg 1$ (deep in the atmosphere). Then

$$I_\nu(0) = \int_0^{\tau_{\nu,\mathrm{max}}} S_\nu(\tau_\nu) e^{-\tau_\nu}\,\mathrm{d}\tau_\nu$$

- note that everything is frequency dependent, as specified by the 'ν' subscripts.

Equation of radiative transfer

$$I_\nu(0) = \int_0^{\tau_{\nu,\max}} S_\nu(\tau_\nu) e^{-\tau_\nu}\, \mathrm{d}\tau_\nu$$

Example. $S_\nu(\tau_\nu)$ = const:

$$\begin{aligned} I_\nu(0) &= S_\nu \int_0^{\tau_{\nu,\max}} e^{-\tau_\nu}\, \mathrm{d}\tau_\nu \\ &= S_\nu \left(1 - e^{-\tau_{\nu,\max}}\right) \\ &\simeq S_\nu \qquad \text{if } \tau_{max} >> 1 \end{aligned}$$

In a real star, $S_\nu(\tau)$ varies with τ.
In fact, in LTE (*Local Thermodynamic Equilibrium*), one finds $S_\nu = B_\nu(T)$, the Planck function. So we need to find the relation between T (temperature) and τ.

The T-τ relation

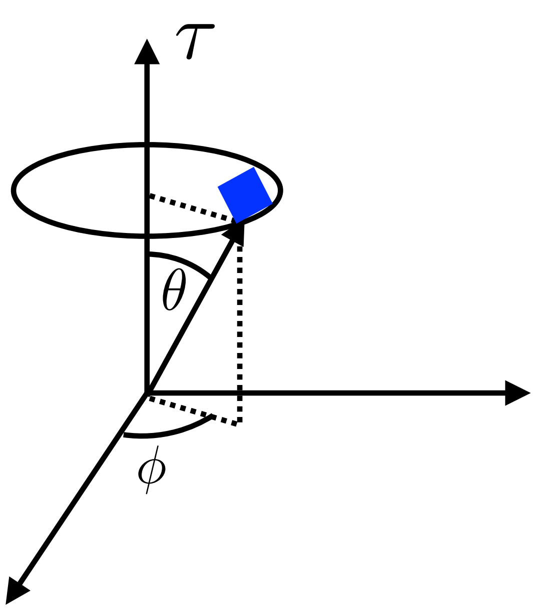
Assumption I: no net energy production in atmosphere. Hence, the flux must be independent of τ.

$$F = \int I \cos\theta \, \mathrm{d}\Omega = \mathrm{const}$$

Assumption II: *I* depends only on θ. Then the flux is

$$F = \int_0^{\pi} \int_0^{2\pi} I \cos\theta \sin\theta \mathrm{d}\phi \mathrm{d}\theta$$

$$= 2\pi \int_0^{\pi} I \cos\theta \sin\theta \mathrm{d}\theta$$



Define $H \equiv \frac{1}{2} \int_{-1}^{1} I(\mu) \mu \, \mathrm{d}\mu$

Flux is $F = 4\pi H$

Moments of the Intensity

$$J = \frac{1}{2}\int_{-1}^{1} I(\mu)\,\mathrm{d}\mu$$

- Mean intensity

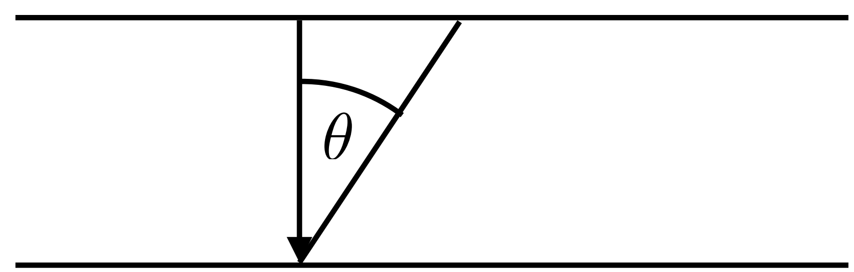
$$H = \frac{1}{2}\int_{-1}^{1} I(\mu)\mu\,\mathrm{d}\mu$$

- Flux

$$K = \frac{1}{2}\int_{-1}^{1} I(\mu)\mu^2\,\mathrm{d}\mu$$

Plane parallel assumption

For most stars, the atmosphere is much thinner than the radius of the star.



From now on, define τ_ν as the optical depth along a line perpendicular to the surface of the star. The Eq. of radiative transfer then becomes

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu$$

$$\mu = \cos\theta$$

for a light ray at an angle θ with respect to the τ “axis”.

T-τ relation cont'd

Multiply Eq. of radiative transfer by μ:

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu$$

$$\mu^2 \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \mu I_\nu - \mu S_\nu$$

Integrate over μ:

$$\int_{-1}^{1} \mu^2 \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} \mathrm{d}\mu = \int_{-1}^{1} \mu I_\nu - \mu S_\nu \mathrm{d}\mu$$

Assume S_ν is independent of μ (second term on r.h.s. disappears):

$$\boxed{\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = H_\nu}$$

$$\begin{aligned} K_\nu &= H_\nu \tau_\nu + \mathrm{const} \\ &= F_\nu \tau_\nu / 4\pi + \mathrm{const} \end{aligned}$$

We then need to find the $T(\tau)$ relation (and hence $\kappa(\tau)$ and $j(\tau)$) that satisfies this equation.

The grey atmosphere

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = H_\nu$$

In general, this equation must be solved numerically. However, under simplifying assumptions, analytic solutions are possible.

Assumption I: κ is independent of ν. Then we can drop the ν subscripts.

Assumption II. Radiation field can be approximated as two uniform "hemispheres" with intensity I_{in} and I_{out}:

$$J(\tau) = \frac{1}{2}\left(I_{\text{in}}(\tau) + I_{\text{out}}(\tau)\right)$$

$$\begin{aligned} K(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\nu)\mu^2\,\mathrm{d}\mu \\ &= \frac{1}{2}\left(I_{\text{in}} + I_{\text{out}}\right)\int_{0}^{1} \mu^2\,\mathrm{d}\mu \\ &= \frac{1}{3}J(\tau) \end{aligned}$$

We thus have the *Eddington approximation*: $K \approx \frac{1}{3}J$

The grey atmosphere

Eddington approximation: $K_\nu \approx \frac{1}{3} J_\nu$

Combining this with $\frac{\mathrm{d}K}{\mathrm{d}\tau} = H$

we get $\frac{\mathrm{d}J}{\mathrm{d}\tau} = 3H$

By assumption, the flux (F=4πH) is constant so $J = 3H\tau + \mathrm{const}$

or $J = 3H(\tau + C)$

The grey atmosphere

$$J = 3H(\tau + C)$$

For a grey atmosphere, the assumption of no energy production/loss also implies *J*=*S*, and in LTE we have

$$J = S = \int S_\nu \mathrm{d}\nu = \int B_\nu \mathrm{d}\nu = \frac{\sigma T^4}{\pi} \qquad \text{(intensity)}$$

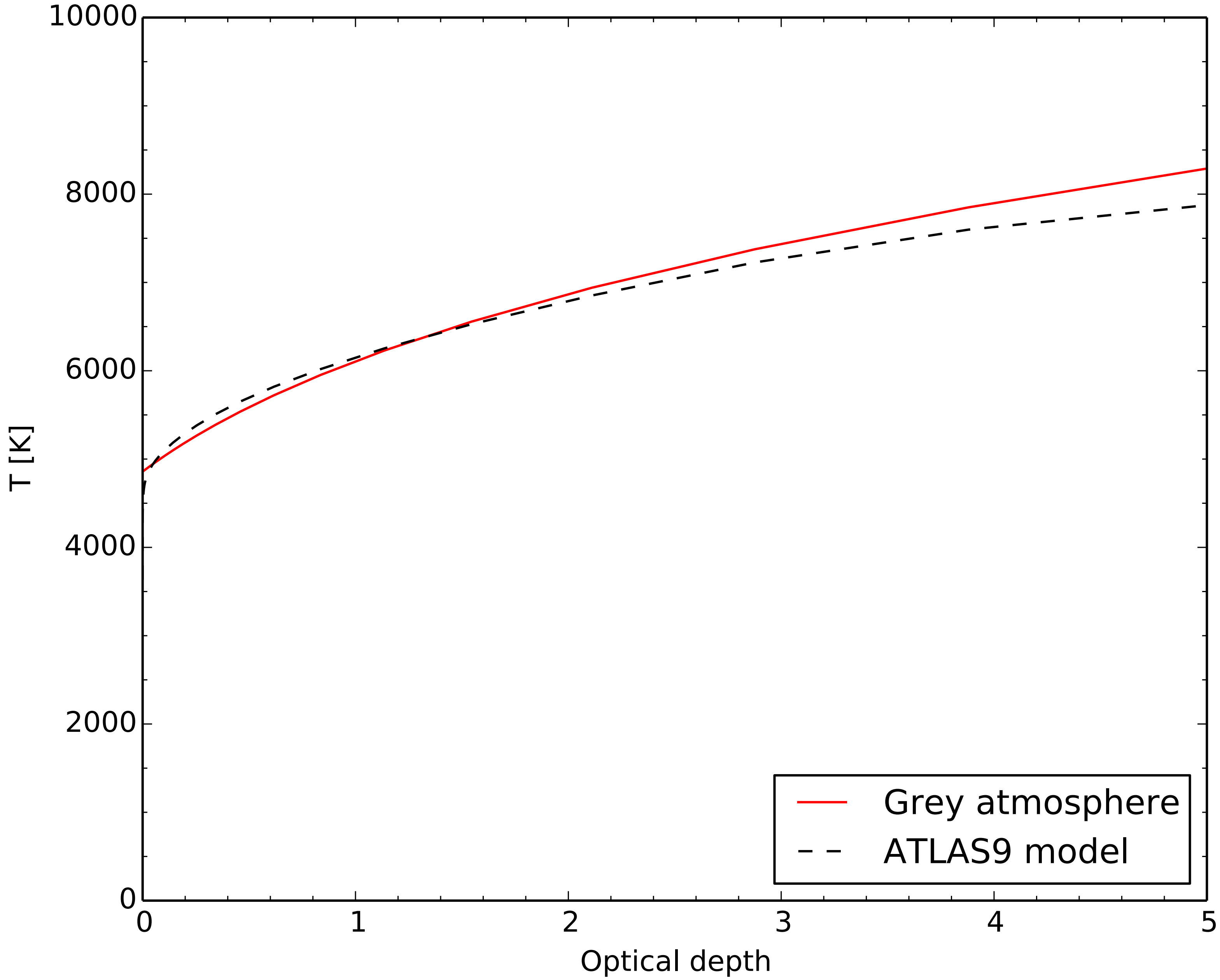
The flux is $F = 4\pi H = \sigma T_{\mathrm{eff}}^4$

So then $\frac{\sigma T^4}{\pi} = \frac{3\sigma}{4\pi} T_{\mathrm{eff}}^4 (\tau + C) \longrightarrow T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 (\tau + C)$

Boundary conditions yield $2\pi J(\tau=0) = F = 4\pi H$, hence $C=2/3$, so

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 (\tau + 2/3)$$

T-τ relation

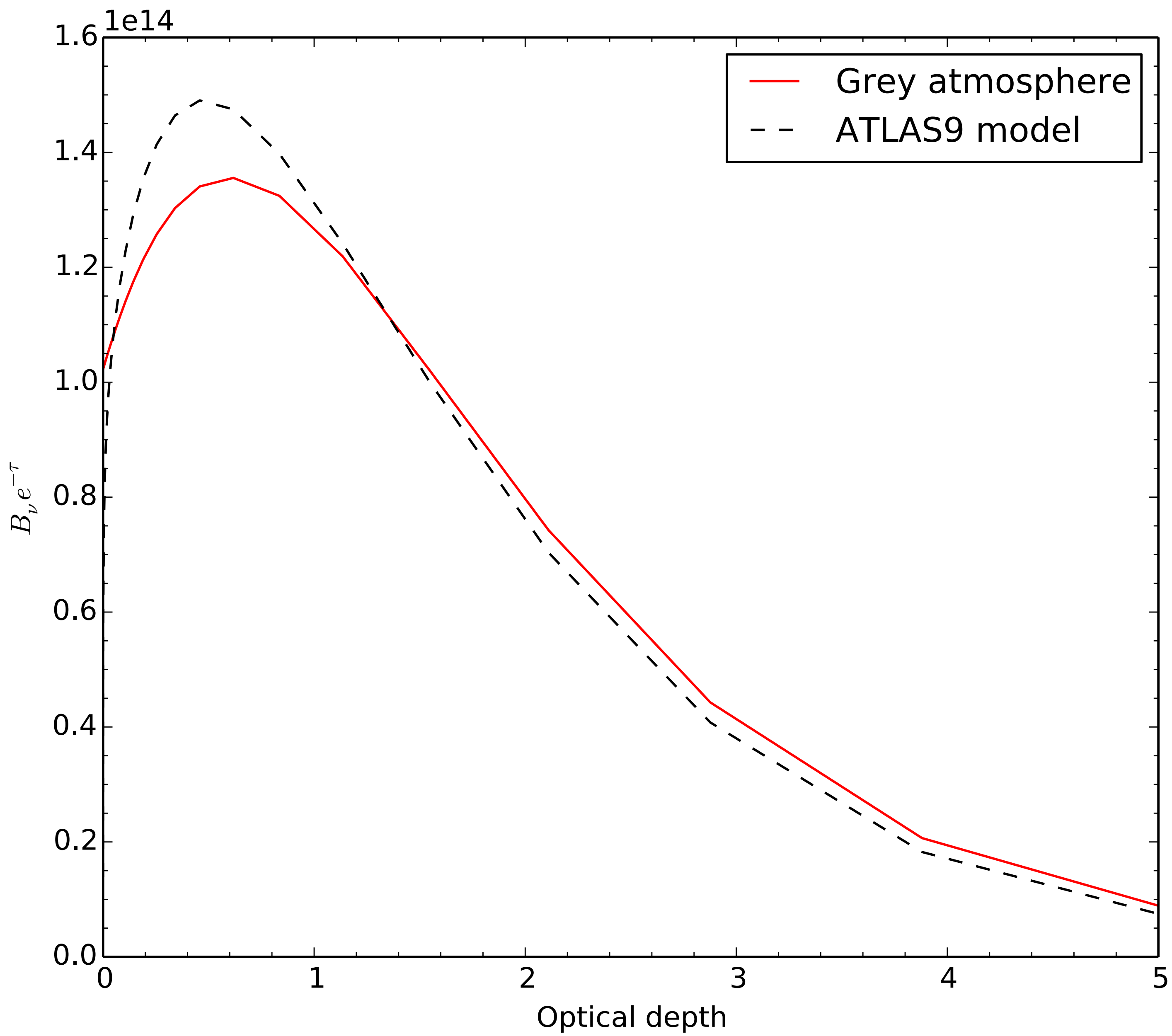


Models for T_{eff}=5777 K

ATLAS9 model:
log g = 4.44 (Sun)
$Z = Z_{\odot}$.

Where does the radiation originate?

Recall solution to eqn. of radiative transfer: $$I_\nu(0) = \int_0^{\tau_{\nu,\mathrm{max}}} S_\nu(\tau_\nu) e^{-\tau_\nu}\, \mathrm{d}\tau_\nu$$



Graph shows the integrand vs. optical depth

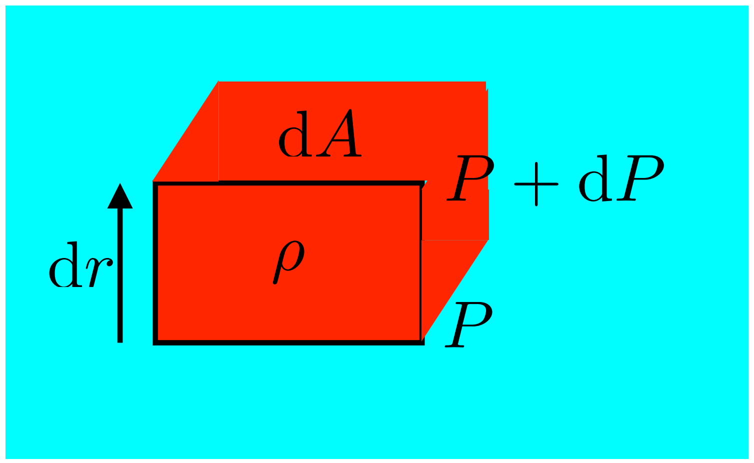
Although *T* increases inwards,

most of the contribution to the observed intensity comes from τ≈1.

Pressure structure

Apply the principle of **hydrostatic equilibrium**. Assume that force of *gravity* is balanced by *pressure gradient* (“buyancy”).

Small element of the atmosphere with area d*A*, thickness d*r*, and density ρ:



Gravity: $F_g = -g\mathrm{d}M = -g\rho\,\mathrm{d}A\,\mathrm{d}r$

Pressure: $F_p = \mathrm{d}P\,\mathrm{d}A$

Equilibrium: $\mathrm{d}P\,\mathrm{d}A = -g\rho\,\mathrm{d}A\,\mathrm{d}r$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$

Divide by $\kappa_0\rho$, where ‘0’ denotes some reference wavelength (e.g. 500 nm):

$$\frac{\mathrm{d}P}{\mathrm{d}\tau_0} = \frac{g}{\kappa_0}$$

Pressure structure

Equation of **hydrostatic equilibrium:**

$$\frac{\mathrm{d}P}{\mathrm{d}\tau_0} = \frac{g}{\kappa_0}$$

Hence, pressure structure can be found by integration

$$P(\tau_0) = g \int_0^{\tau_0} \frac{\mathrm{d}\tau_0'}{\kappa_0(\tau_0')}$$

But κ will depend on both *P* and *T*, so the equation must be solved iteratively (together with the T(τ) relation).

P is here the *total* pressure (gas + radiation + turbulence + …). Usually P_{gas} dominates.

Model atmospheres in practice

- Commonly used "standard codes":
  `ATLAS9`, `ATLAS12` (by R. Kurucz)
  `MARCS` (Uppsala group)

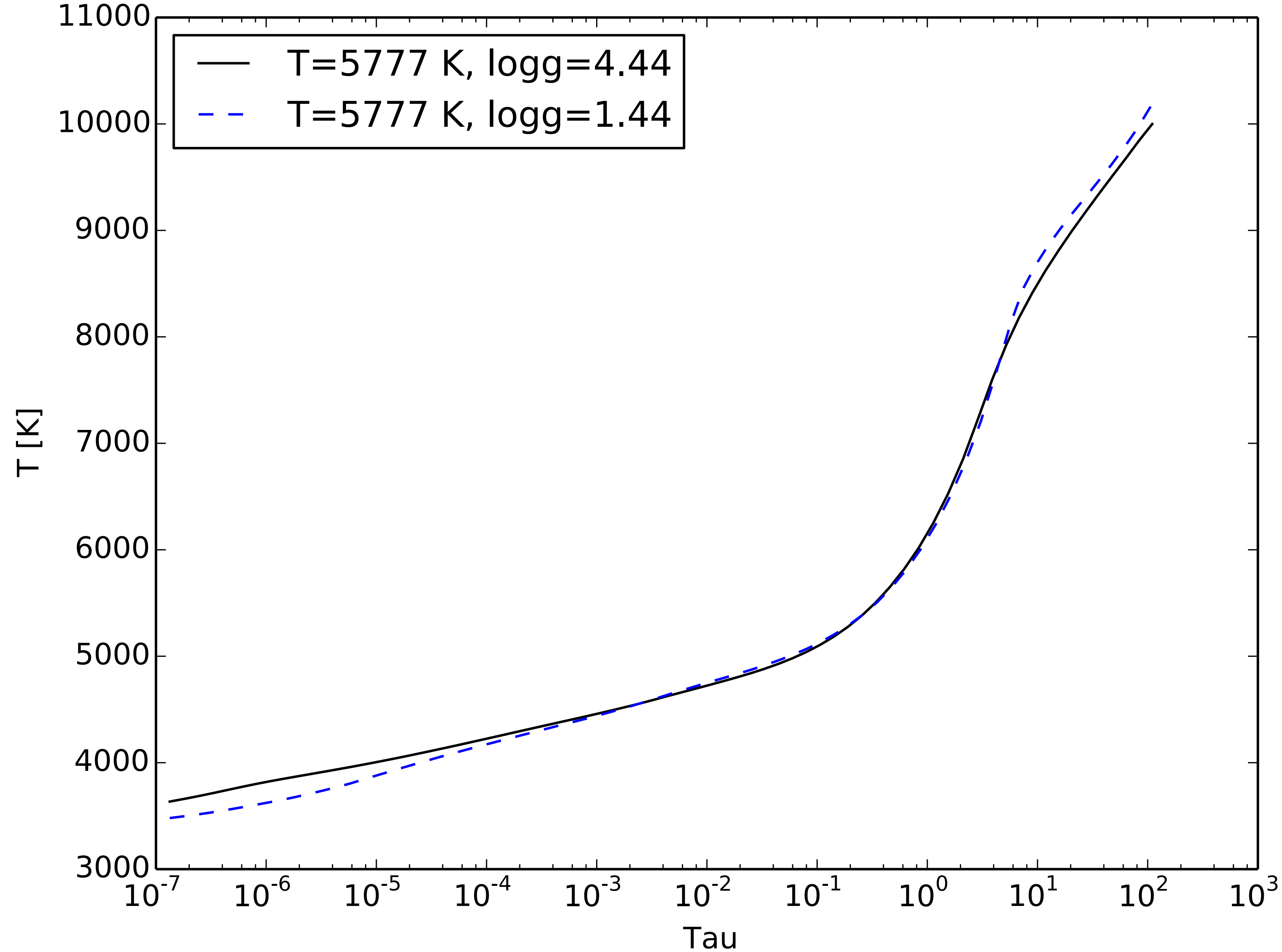
- Both codes assume LTE, steady-state. One-dimensional (physical quantities only vary in the vertical direction)

- `ATLAS` models assume plane-parallel geometry; `MARCS` models are available for spherical geometry

- `ATLAS` (Fortran) codes are publicly available; Running a model takes a few sec on a modern PC.
  `MARCS` models can be downloaded via website.

Model atmospheres - examples

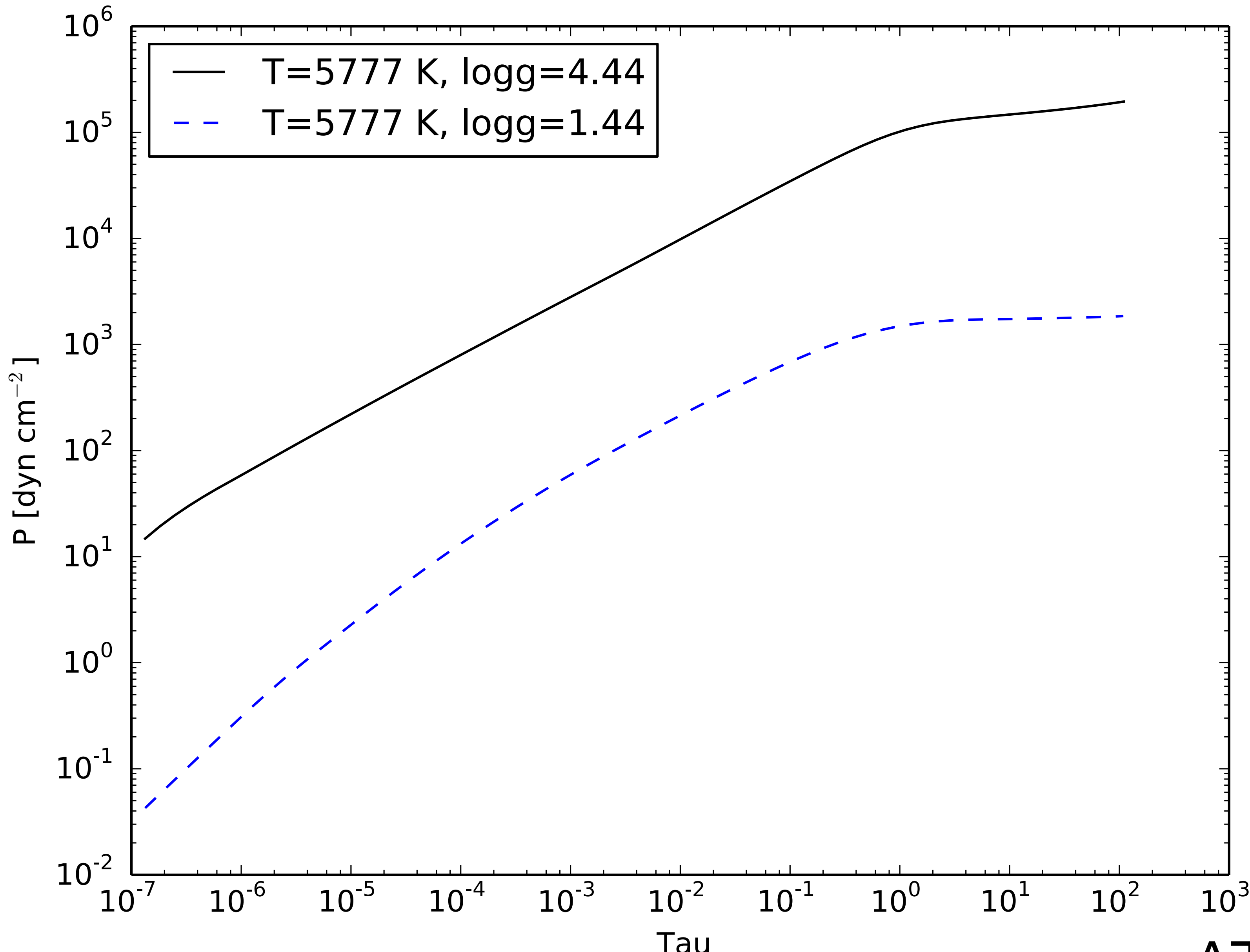
T-τ relations (Sun and Yellow Supergiant)



ATLAS9 models

Model atmospheres - examples

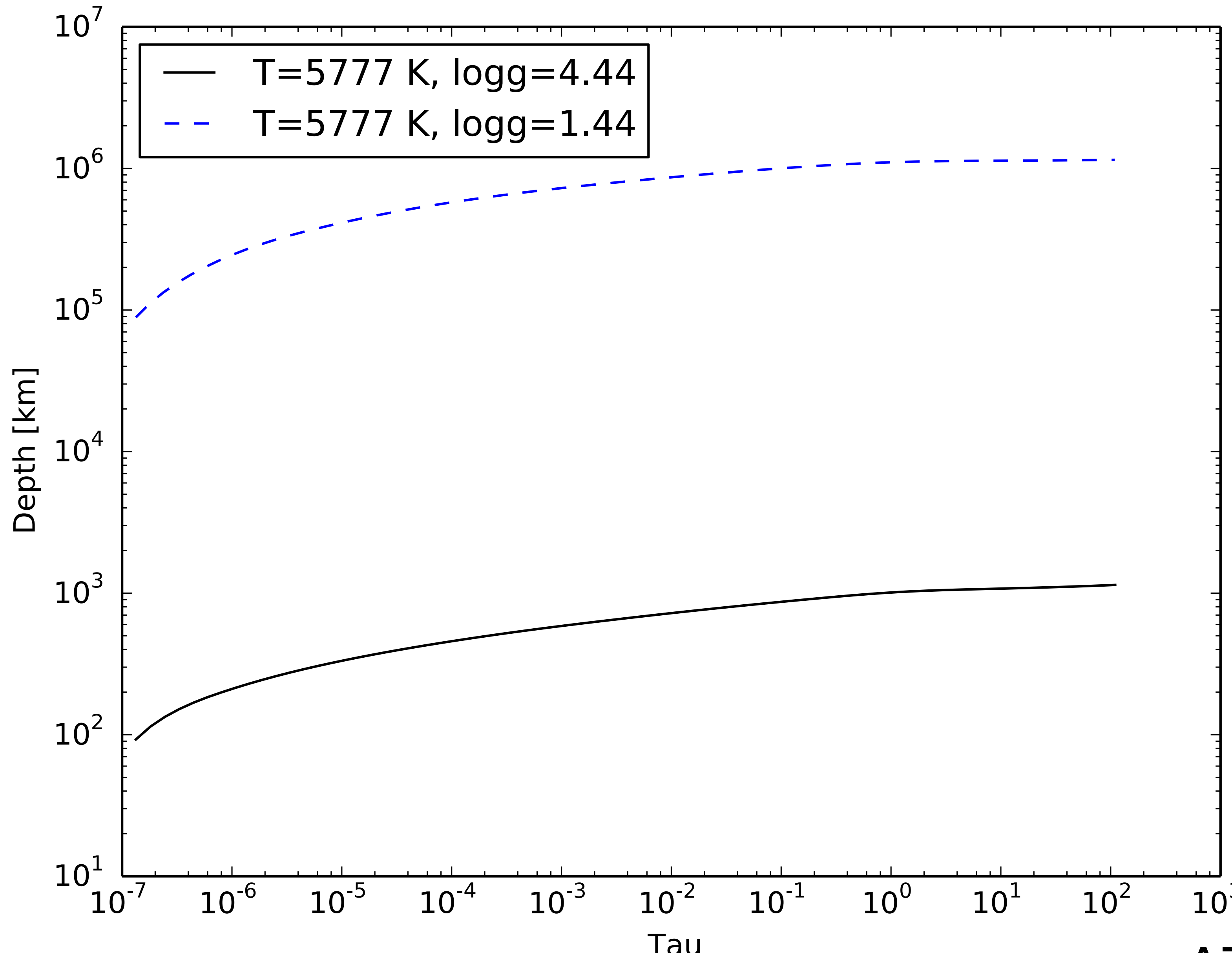
P-τ relations (Sun and Yellow Supergiant)



ATLAS9 models

Model atmospheres - examples

Physical depth vs τ



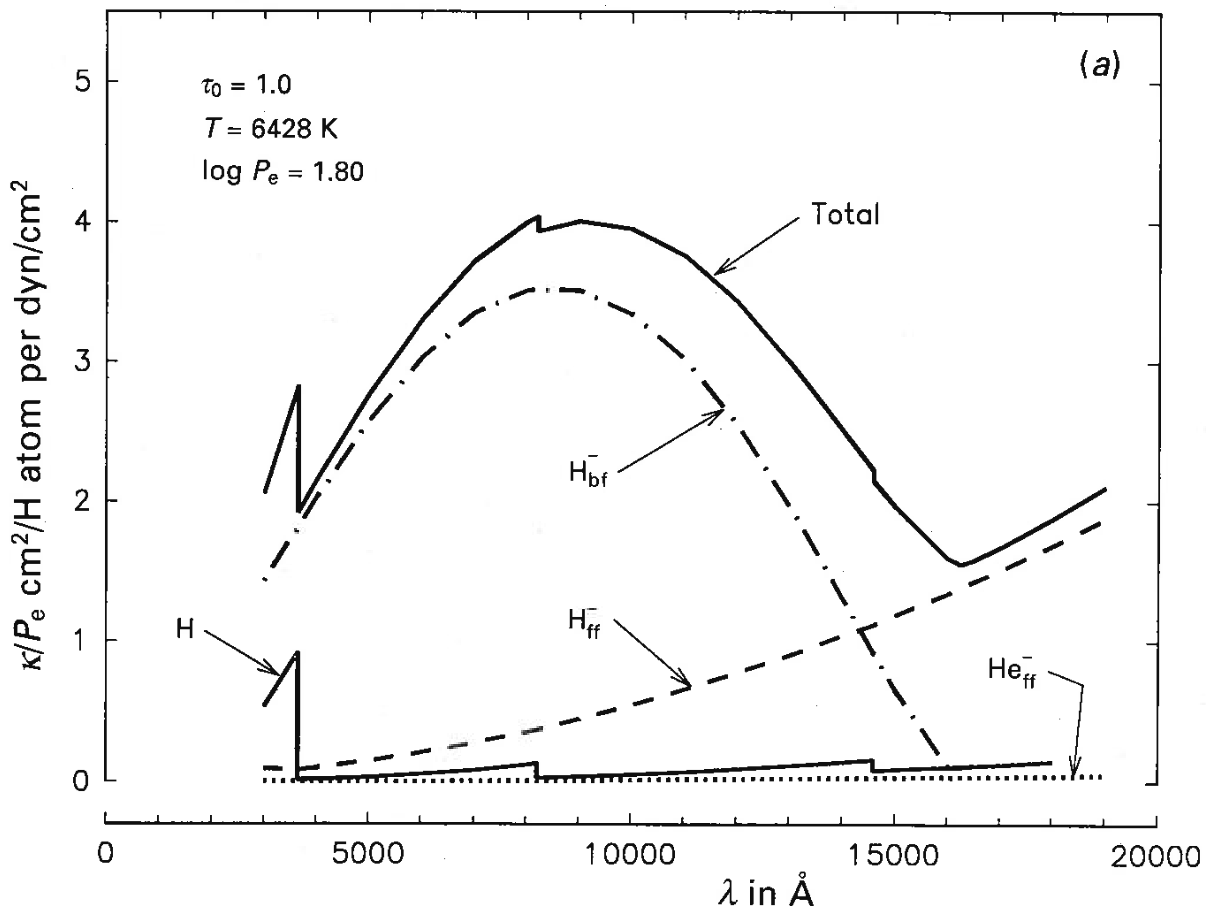
ATLAS9 models

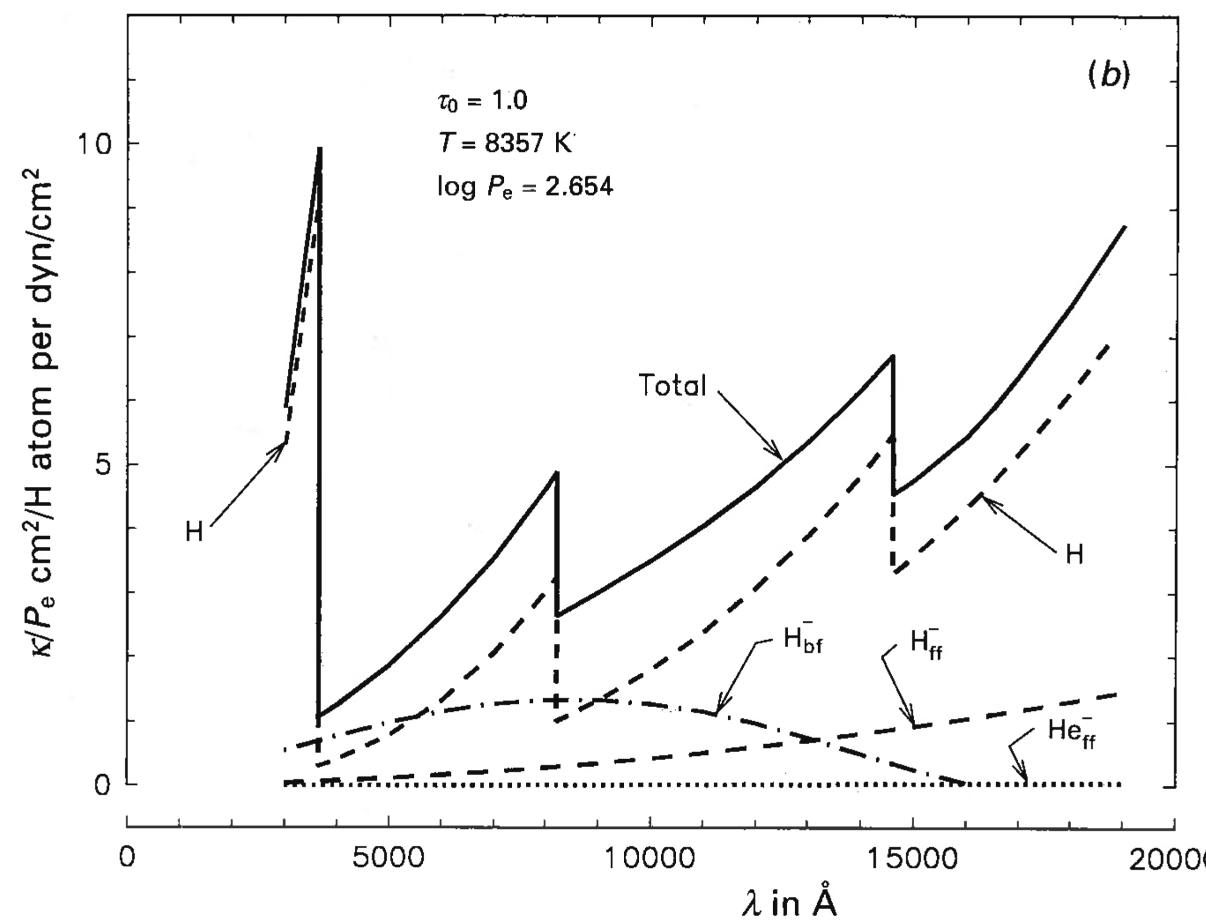
Opacity sources

Opacity sources

- Continuum opacity: bound-free and free-free processes, mainly from H and H^-
- In the Sun, the main continuum opacity source is bound-free transitions in H^-; in hotter stars H becomes dominant.
- Line opacity: bound-bound transitions in atoms (and molecules, in cool stars)

Continuum opacity





D. F. Gray *The Observation and Analysis of Stellar Photospheres*

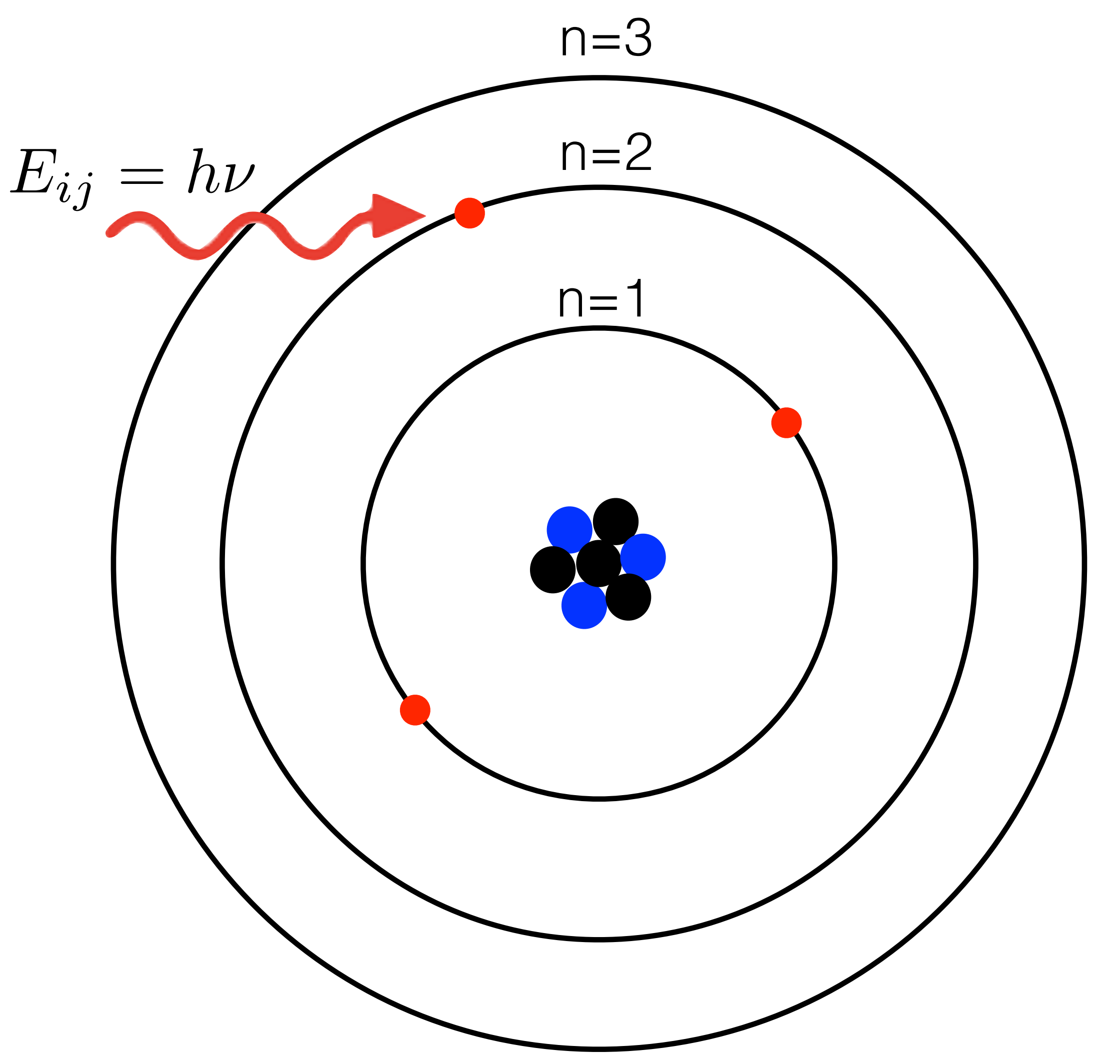
Line opacity

To calculate this, we need to calculate:

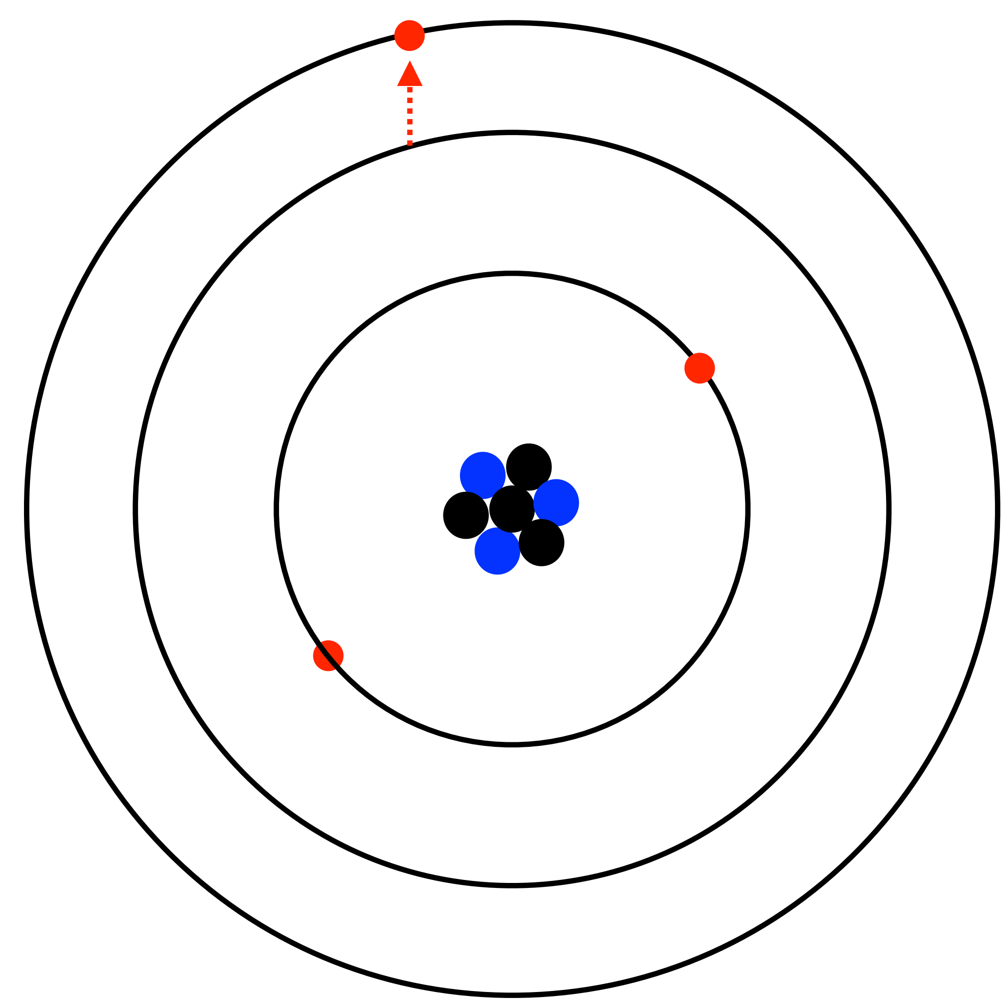
- Number of atoms in the relevant state of *ionization* (Saha's equation)
- Fraction of those atoms that are in the relevant state of *excitation* (Boltzmann's equation)
- Probabilities that a transition from one energy level to another will occur, so that a photon is absorbed or emitted (Einstein coefficients or oscillator strengths)

Absorption

(1): Atom in energy level i +
photon with energy $E_{ij} = h\nu$

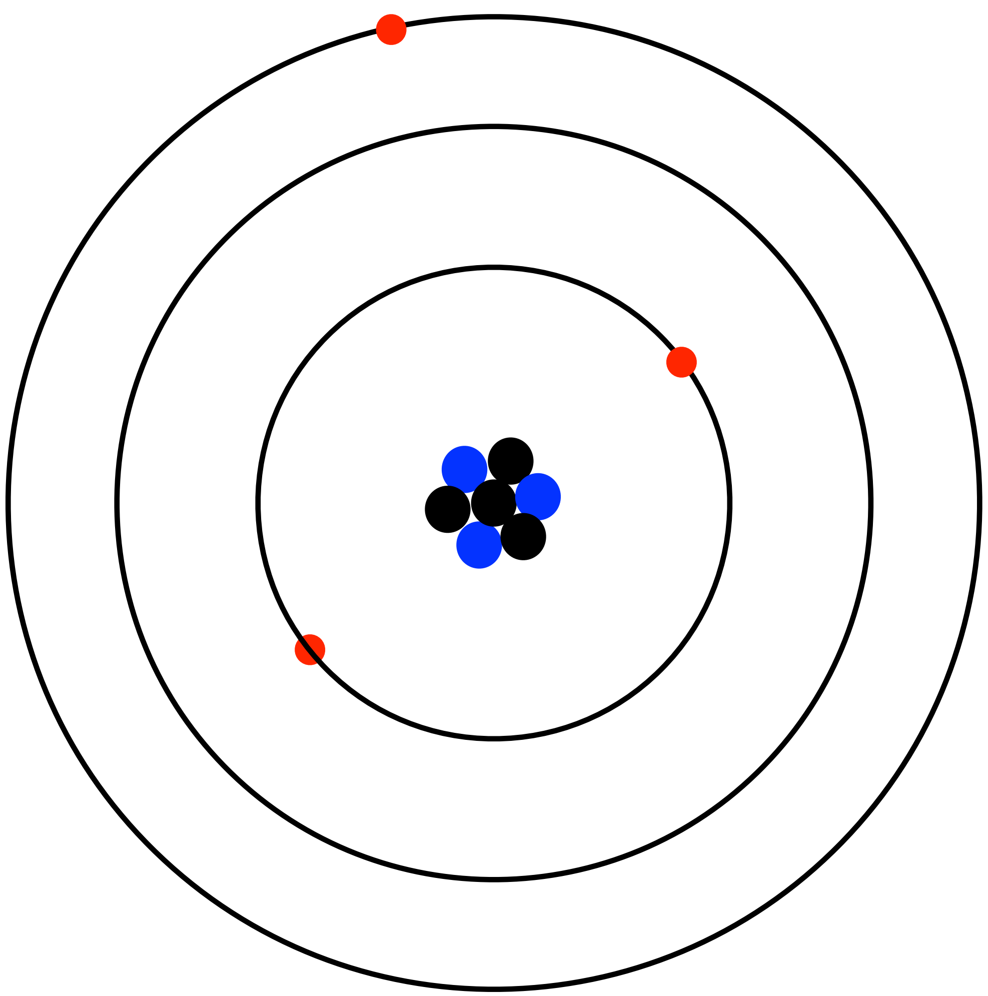


(2): Atom in energy level j,
$E_j = E_i + h\nu$

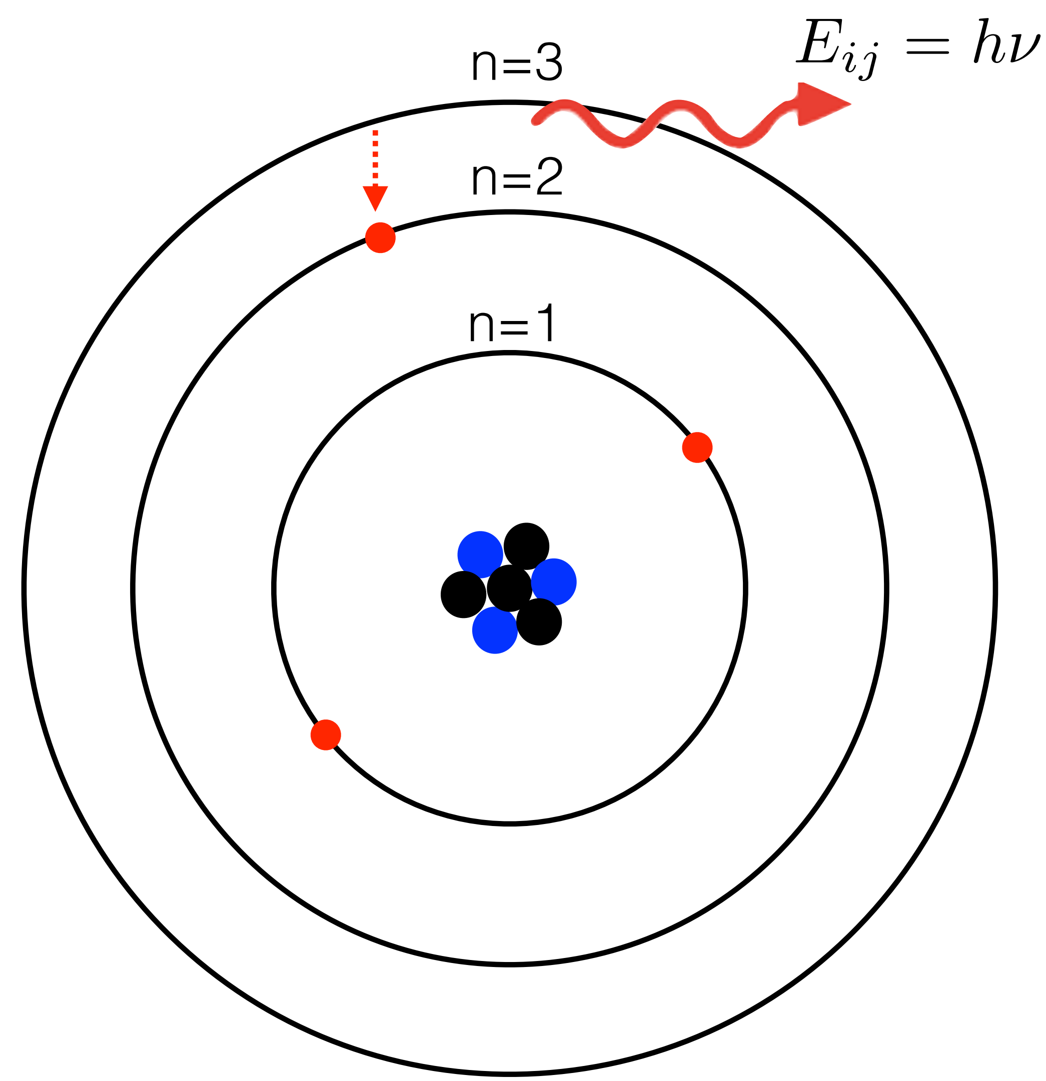


Spontaneous emission

(1): Atom in energy level *j*

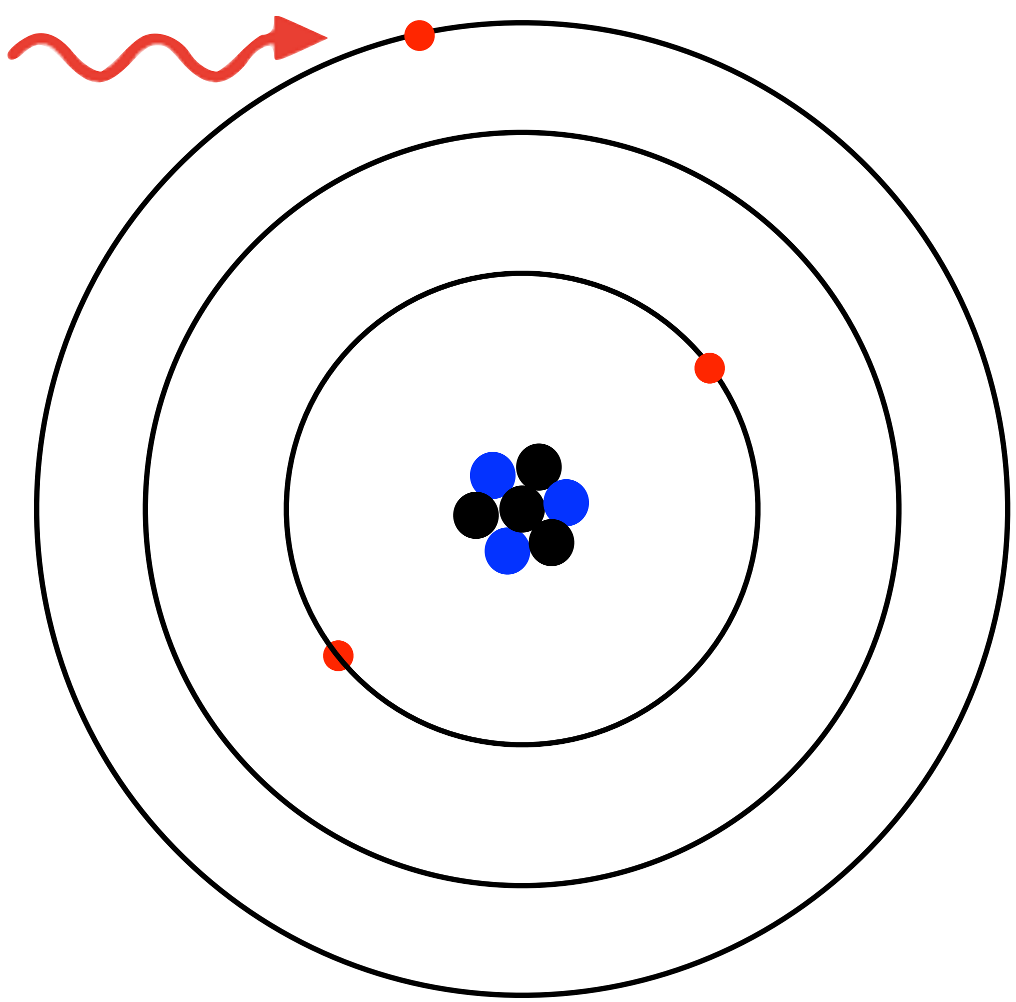


(2): Atom in energy level *i* +
photon with energy E_{ij} = $h\nu$

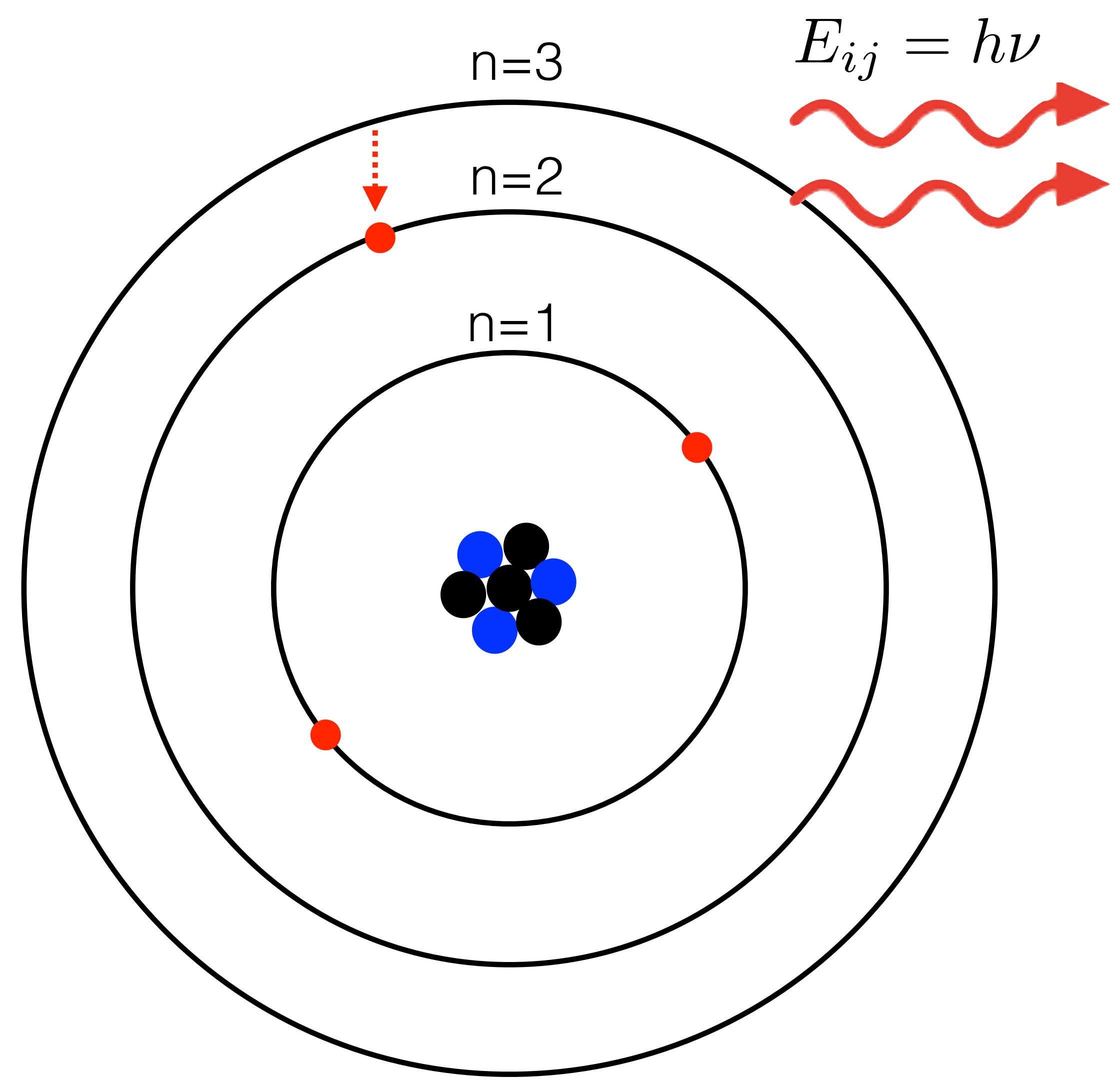


Stimulated emission

(1): Atom in energy level *j*,
+ photon, $E_{ij} = E_i + h\nu$



(2): Atom in energy level *i* +
extra photon



Transition probabilities

Spontaneous emission: $P_{ji} = A_{ji}$.

P_{ji} = Probability of decay from level j to level i per unit time
A_{ji} = Einstein coefficient for spontaneous emission

Absorption: $P_{ij} = 4 \pi J_{\nu} B_{ij}$.

P_{ij} = Probability (per unit time) that a photon is absorbed so a transition from level i to level j occurs
J_{ν} = Mean intensity of radiation field at frequency ν
B_{ij} = Einstein coefficient for absorption

Stimulated emission: $P_{ji} = 4 \pi J_{\nu} B_{ji}$.

P_{ji} = Probability (per unit time) that a photon stimulates decay from level j to level i and emission
J_{ν} = Mean intensity of radiation field at frequency ν
B_{ji} = Einstein coefficient for stimulated emission

Relations between Einstein coefficients

Stimulated vs. spontaneous emission:

$$A_{ji} = (8\pi h\nu^3/c^2)B_{ji}$$

Stimulated emission vs. absorption:

$$B_{ij} = (g_j/g_i)B_{ji}$$

g_j and g_i are the *statistical weights* of the energy levels (e.g. Hydrogen: g(n=1)=2, g(n=2)=8, etc..)

In stellar spectroscopy, we often use *oscillator strengths*, *f*-values:

$$f = \frac{4\pi\epsilon_0 mc^3}{8\pi^2 e^2}\frac{1}{\nu^2}\frac{g_j}{g_i}A_{ji} = 1.347\times 10^{21}\frac{1}{\nu^2}\frac{g_j}{g_i}A_{ji}$$

Line emission and absorption coefficients

Emission (into one unit solid angle) per unit volume:

$$j_\nu \rho = \frac{1}{4\pi} N_j A_{ji} h\nu$$

for N_j atoms per unit volume in level j

Absorption per unit volume:

$$\kappa_\nu \rho = \left(N_i B_{ij} - N_j B_{ji}\right) h\nu$$

for
N_i atoms per unit volume in level i and
N_j atoms per unit volume in level j

Note: stimulated emission is treated as
"negative absorption" !

The source function

Recall definition of source function:

$$S_\nu = \frac{j_\nu \rho}{\kappa_\nu \rho} = \frac{\frac{1}{4\pi} N_j A_{ji} h\nu}{(N_i B_{ij} - N_j B_{ji})\, h\nu}$$

Divide through by $N_j\ B_{ji}\ h\ \nu$:

$$S_\nu = \frac{\frac{1}{4\pi} A_{ji}/B_{ji}}{\left(\frac{N_i}{N_j} B_{ij}/B_{ji} - 1\right)}$$

Use the relations between the Einstein coefficients:

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{N_i g_j}{N_j g_i} - 1}$$

The source function

General expression for source function:

$$S_\nu = \frac{2h\nu^3/c^2}{\frac{N_i g_j}{N_j g_i} - 1}$$

In local thermodynamic equilibrium (LTE), we have the Boltzmann eq.:

$$\frac{N_j}{N_i} = \frac{g_j}{g_i} e^{-E_{ij}/kT} = \frac{g_j}{g_i} e^{-h\nu/kT}$$

So in LTE the source function is

$$S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

$= B_\nu$,
the Planck function

Line profiles

Spectral lines are not infinitely “sharp” (δ-functions).

They are broadened by

- Doppler broadening (motions of atoms)
- Natural broadening (finite lifetimes of energy levels)
- Pressure broadening (collisions between atoms)

Doppler broadening

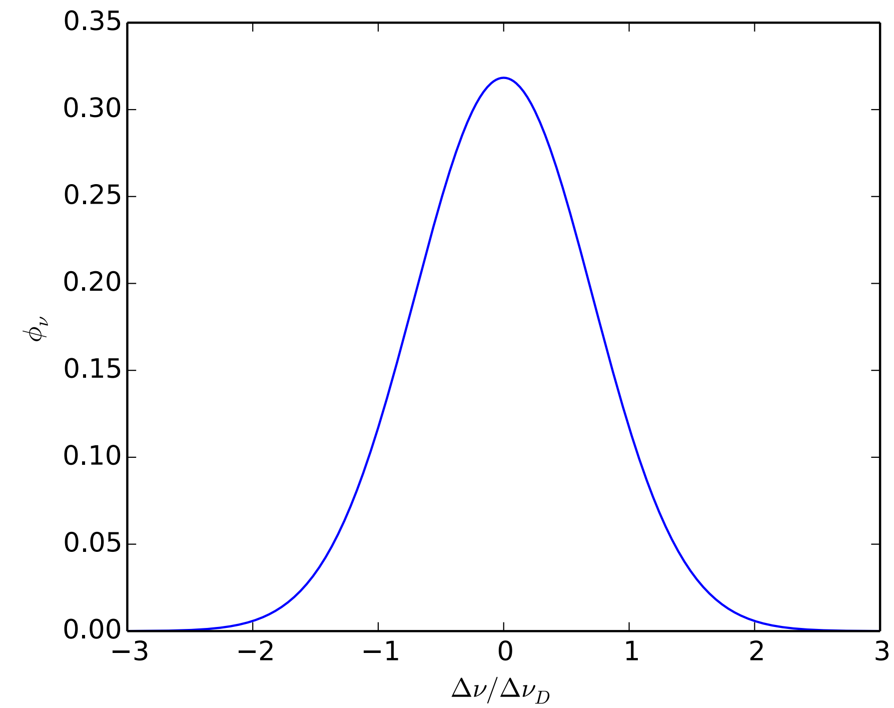
Frequency shift for source moving at velocity, v:

$$\frac{\Delta\nu}{\nu} = -\frac{v}{c}$$

For temperature *T*, this leads to *Gaussian* line profiles

$$\phi_\nu = \frac{1}{\pi \Delta\nu_D} e^{-(\Delta\nu/\Delta\nu_D)^2}$$

$$\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$$



In addition to thermal motions, we may include a contribution from *microturbulence*:

$$\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m} + V_t^2}$$

Natural and pressure broadening

Natural broadening:
According to Heisenberg's uncertainty principle, we have

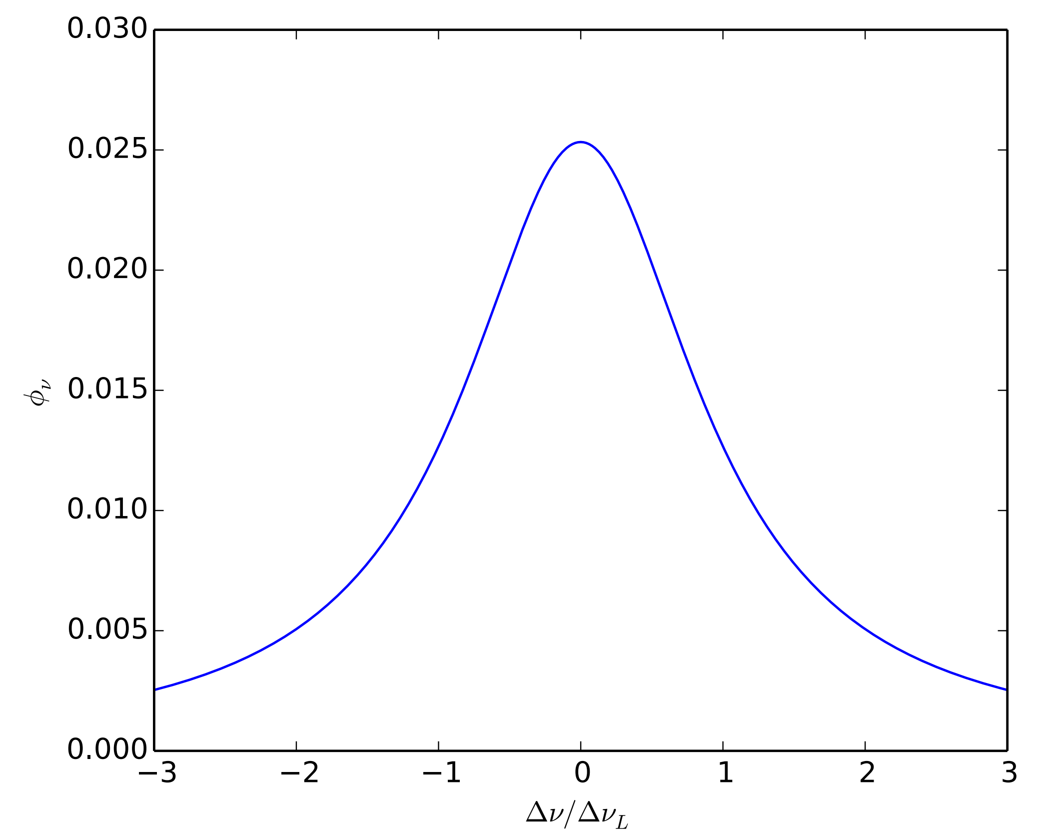
$$\Delta E \Delta t \approx h/2\pi$$

If the mean lifetime of an energy level is Δt, we thus expect a line broadening of

$$\Delta E \approx \frac{h}{2\pi \Delta t}$$

The line profile is given by the *Lorentzian* form:

$$\phi_\nu = \frac{A_{ij}}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + (A_{ji}/4\pi)^2}$$



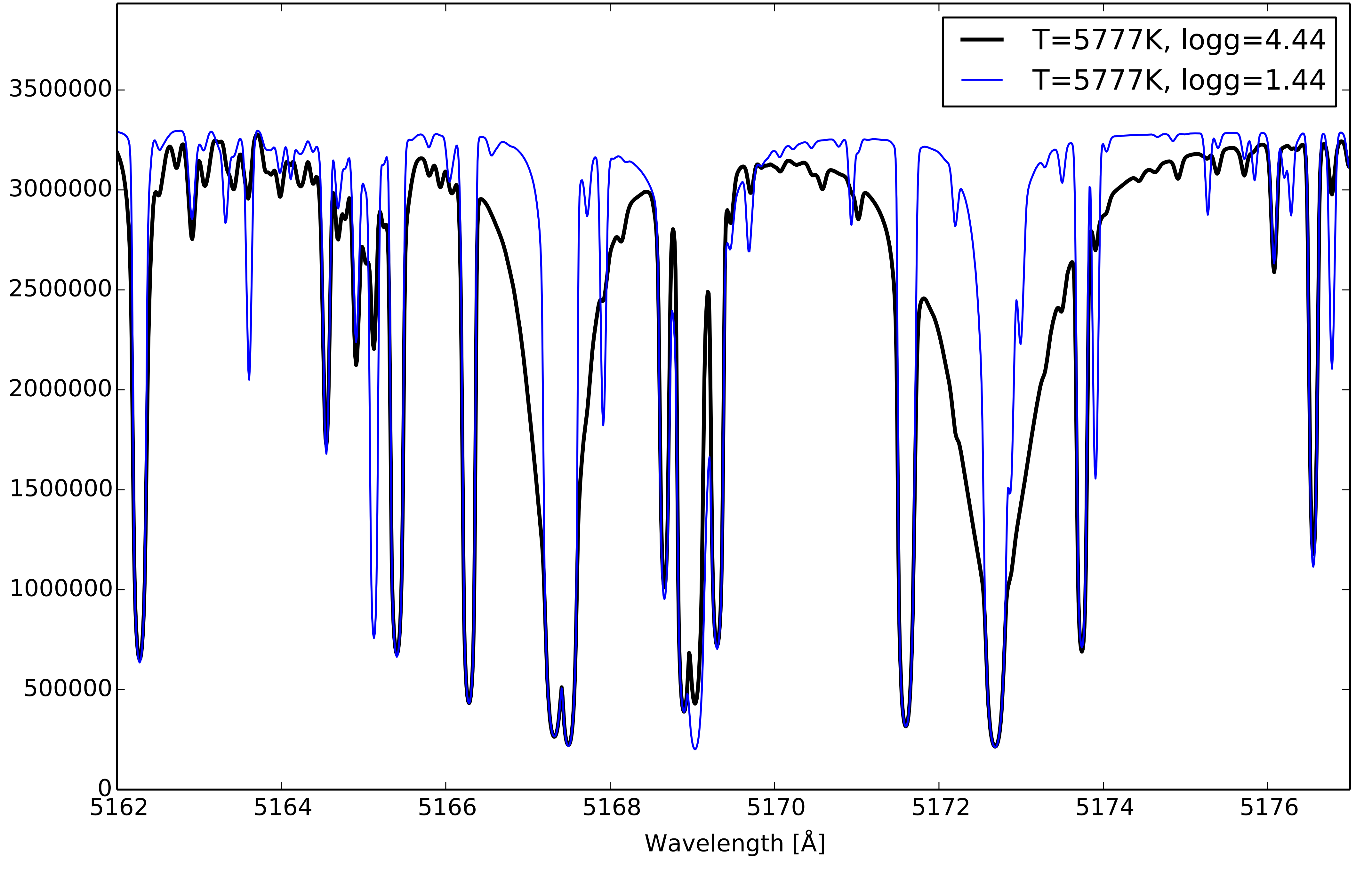
Natural and pressure broadening

Pressure broadening:
Profiles are again Lorentzian, similar to natural broadening

$$\phi_\nu = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2}$$

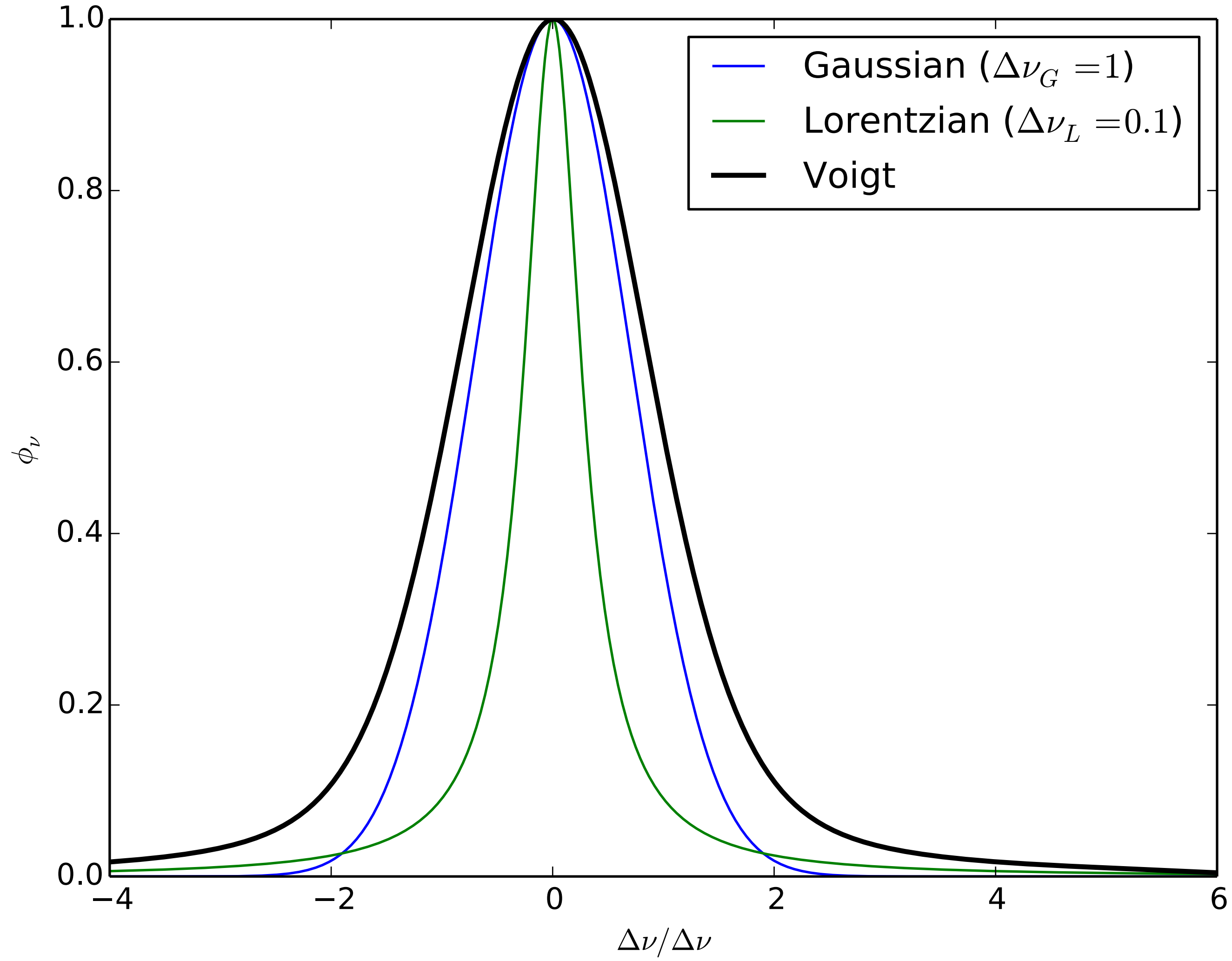
Here, $\mathbf{\Gamma}$ is (roughly) the average interaction rate.

Higher density/pressure → more interactions → broader lines


T=5777K, logg=4.44
T=5777K, logg=1.44
3500000
3000000
2500000
2000000
1500000
1000000
500000
0
5162
5164
5166
5168
5170
5172
5174
5176
Wavelength [Å]

The Voigt profile

Real line profiles are a convolution of the Gaussian and Lorentzian profiles. The result is called a *Voigt* profile:



Note that the power-law wings of the Lorentzian component always dominate far from the line centre.

Formation of spectral lines

Back to the formal solution to the equation of radiative transfer:

$$I_\nu(0) = \int_0^\infty S_\nu(\tau_\nu) e^{-\tau_\nu} \, \mathrm{d}\tau_\nu$$

Usually, one uses a “standard” optical depth, τ_s, (e.g. at 500 nm)

The absorption coefficient is the sum of line- and continuum absorption,

$$\kappa_\nu = \kappa_{\nu,L} + \kappa_C$$

and

$$\frac{\mathrm{d}\tau_\nu}{\mathrm{d}\tau_s} = \frac{\kappa_\nu}{\kappa_s} = \frac{\kappa_{\nu,L} + \kappa_C}{\kappa_s}$$

Hence, the intensity at a given frequency is found from

$$I_\nu = \int_0^\infty S_\nu(\tau_\nu[\tau_s]) \frac{\kappa_{\nu,L} + \kappa_C}{\kappa_s} e^{-\tau_\nu[\tau_s]} \, \mathrm{d}\tau_s$$

Formation of spectral lines

Hence, the intensity at a given frequency is found from

$$I_\nu = \int_0^\infty S_\nu(\tau_\nu\,[\tau_s])\,\frac{\kappa_{\nu,L} + \kappa_C}{\kappa_s}\,e^{-\tau_\nu\,[\tau_s]}\,\mathrm{d}\tau_s$$

In LTE, we have $S_\nu = B_\nu(\tau_\nu)$, which follows from the T-τ relation.

Note that the LTE assumption may not be valid!
But if we assume that it is, then things are much easier.

Depth of spectral lines

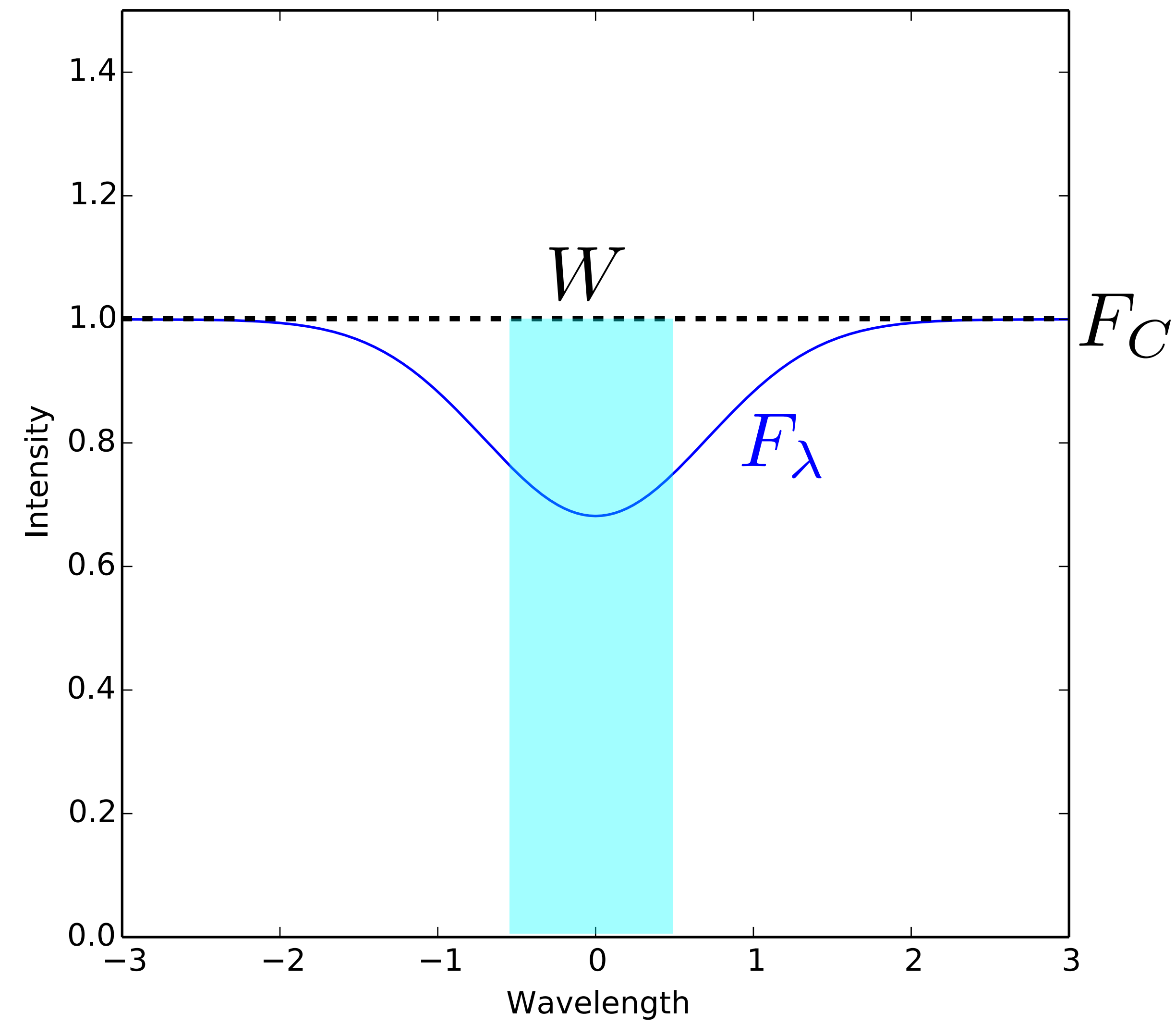
Equivalent width:

$$W = \int \text{"depth"}\, \mathrm{d}\lambda$$

Generally, F_C~const across a line, so

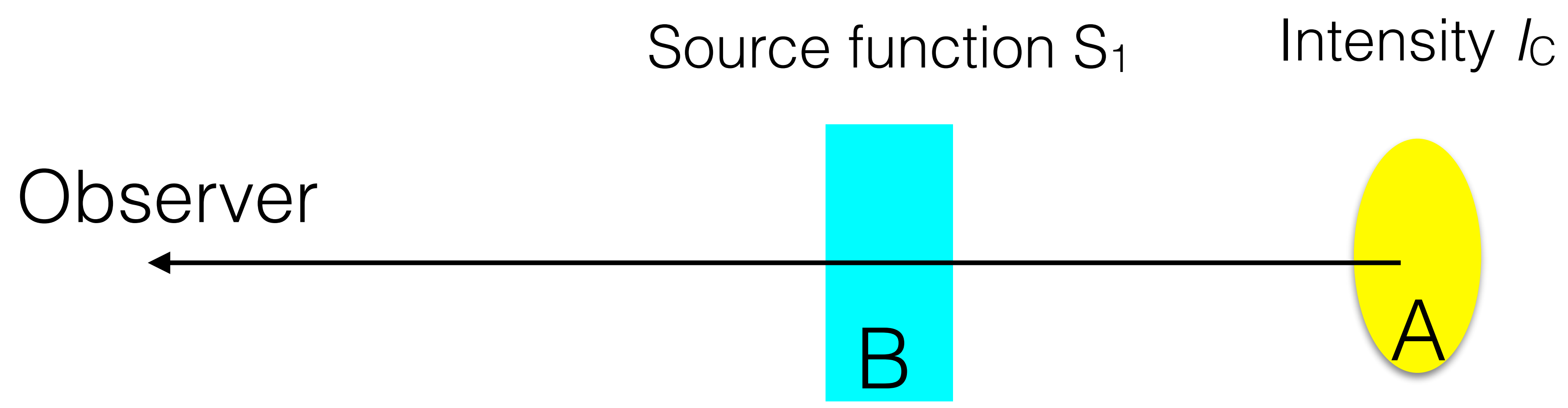
$$W = \int \frac{F_C - F(\lambda)}{F_C} \mathrm{d}\lambda$$

$$= \frac{\lambda^2}{c} \int \frac{F_C - F(\nu)}{F_C} \mathrm{d}\nu$$



Equivalent widths

Example: light from source at **A**, passing through cloud at **B**.
Cloud has source function S_1 at frequency v.



Observed continuum intensity = I_C.

Observed line intensity:

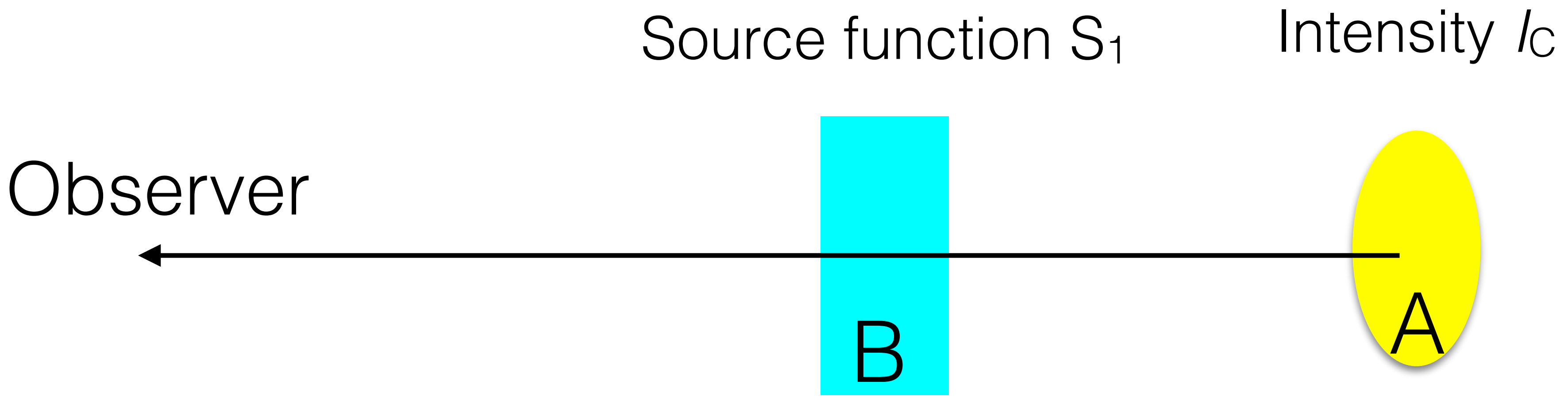
$$I_\nu = \int_0^{\tau_\nu} S_1 e^{-t_\nu} \mathrm{d}t_\nu + I_C e^{-\tau_\nu}$$

$$= S_1(1 - e^{-\tau_\nu}) + I_C e^{-\tau_\nu}$$

The equivalent width is:

$$W = \frac{\lambda^2}{c} \int \frac{I_C - I_\nu}{I_C} \mathrm{d}\nu$$

Equivalent widths



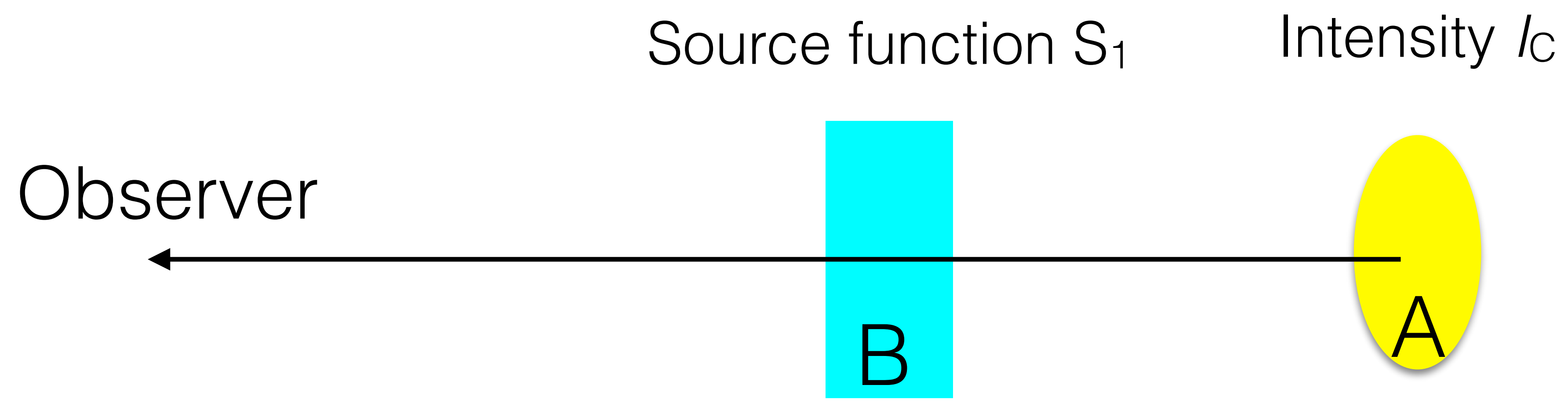
Line intensity: $$I_\nu = S_1(1 - e^{-\tau_\nu}) + I_C e^{-\tau_\nu}$$

The equivalent width: $$W = \frac{\lambda^2}{c} \int \frac{I_C - I_\nu}{I_C} \mathrm{d}\nu$$

$$= \frac{\lambda^2}{c} \int \frac{I_C - S_1(1 - e^{-\tau_\nu}) - I_C e^{-\tau_\nu}}{I_C} \mathrm{d}\nu$$

$$= \frac{\lambda^2}{c} \frac{I_C - S_1}{I_C} \int (1 - e^{-\tau_\nu}) \mathrm{d}\nu$$

Equivalent widths



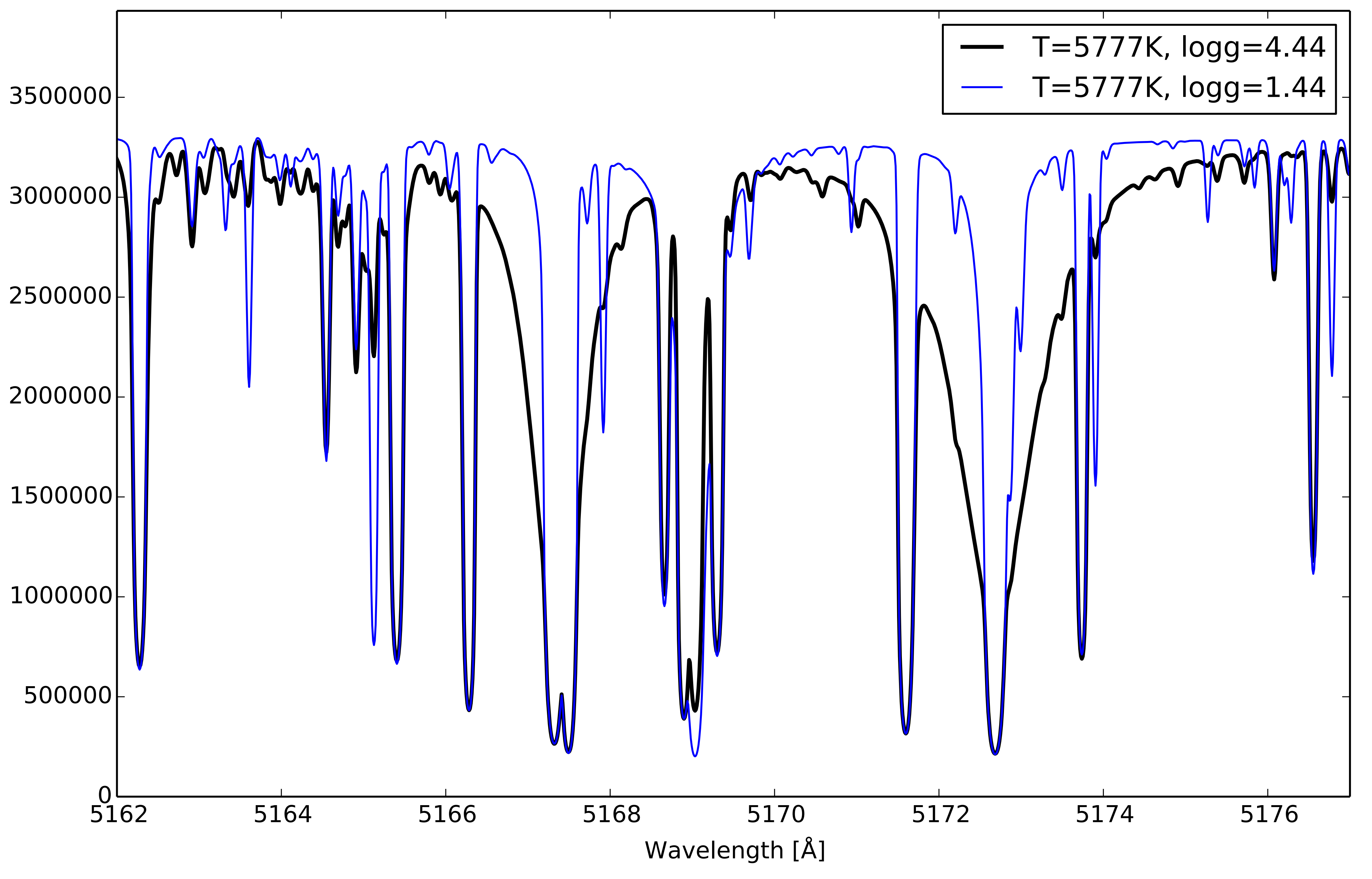
The equivalent width: $$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\int(1 - e^{-\tau_\nu})\mathrm{d}\nu$$

The depth at frequency v is: $$r_\nu = \left(1 - \frac{S_1}{I_C}\right)(1 - e^{-\tau_\nu})$$

so even if the line is very strong, there is a maximum depth:

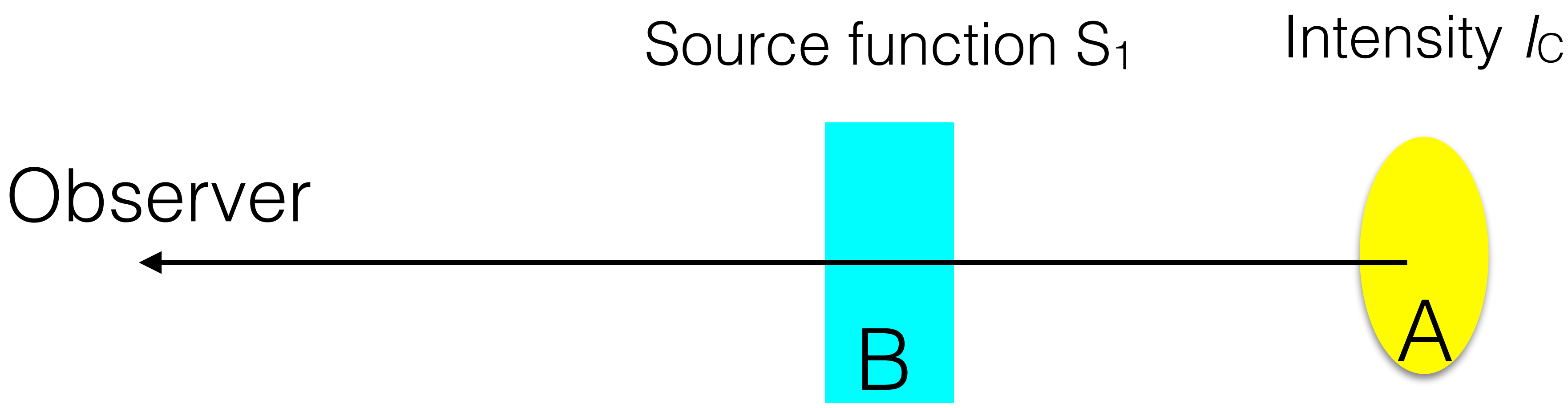
$$r_{\max} = \left(1 - \frac{S_1}{I_c}\right)$$

- the line centre is never completely "dark" (as long as $S_1>0$)



T=5777K, logg=4.44
T=5777K, logg=1.44
0
500000
1000000
1500000
2000000
2500000
3000000
3500000
5162
5164
5166
5168
5170
5172
5174
5176
Wavelength [Å]

Equivalent widths



The equivalent width: $$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\int (1 - e^{-\tau_\nu})\mathrm{d}\nu$$

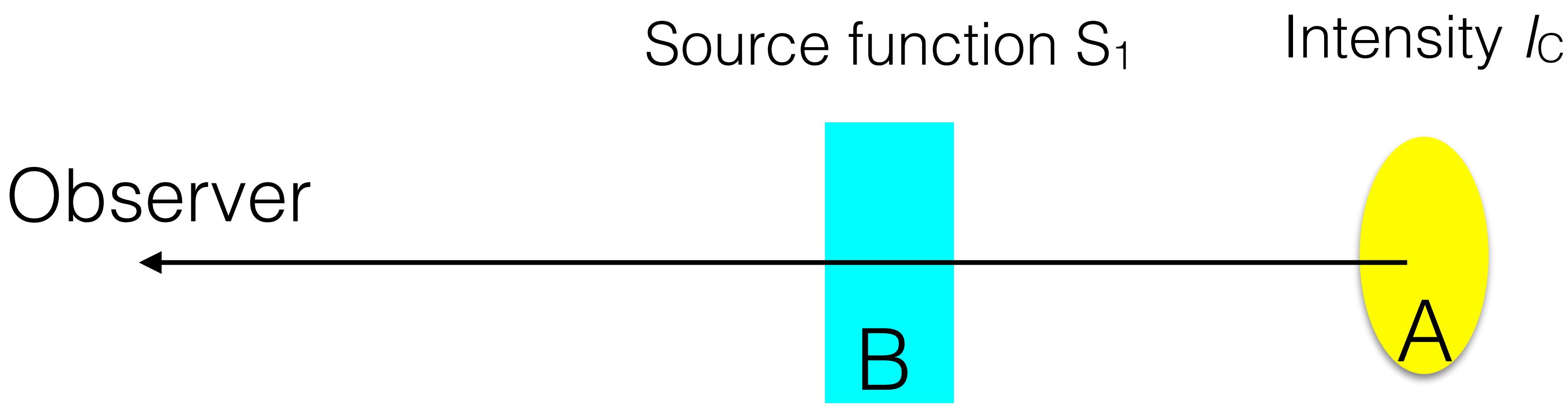
For a weak line ($\tau_\nu \ll 1$), $1 - e^{-\tau_\nu} \approx 1 - (1 - \tau_\nu) = \tau_\nu$

so $$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\int \tau_\nu \mathrm{d}\nu$$

Also, we have $\tau_\nu = \kappa_\nu \rho h$ (for cloud thickness and density h, ρ) so

$$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\rho h \int \kappa_\nu \mathrm{d}\nu$$

Equivalent widths



$$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\rho h \int \kappa_\nu \mathrm{d}\nu$$

Writing $\kappa_\nu = \kappa_0 \phi(\nu)$

$$= N_{\mathrm{abs}}\sigma_0\phi(\nu)/\rho \qquad = N_{\mathrm{abs}} f \frac{\pi e^2}{4\pi\epsilon_0 mc}\phi(\nu)/\rho$$

and remembering $\int \phi(\nu)\mathrm{d}\nu \equiv 1$

we get (for weak lines) $W \propto h N_{\mathrm{abs}} f \frac{\pi e^2}{4\pi\epsilon_0 mc}$

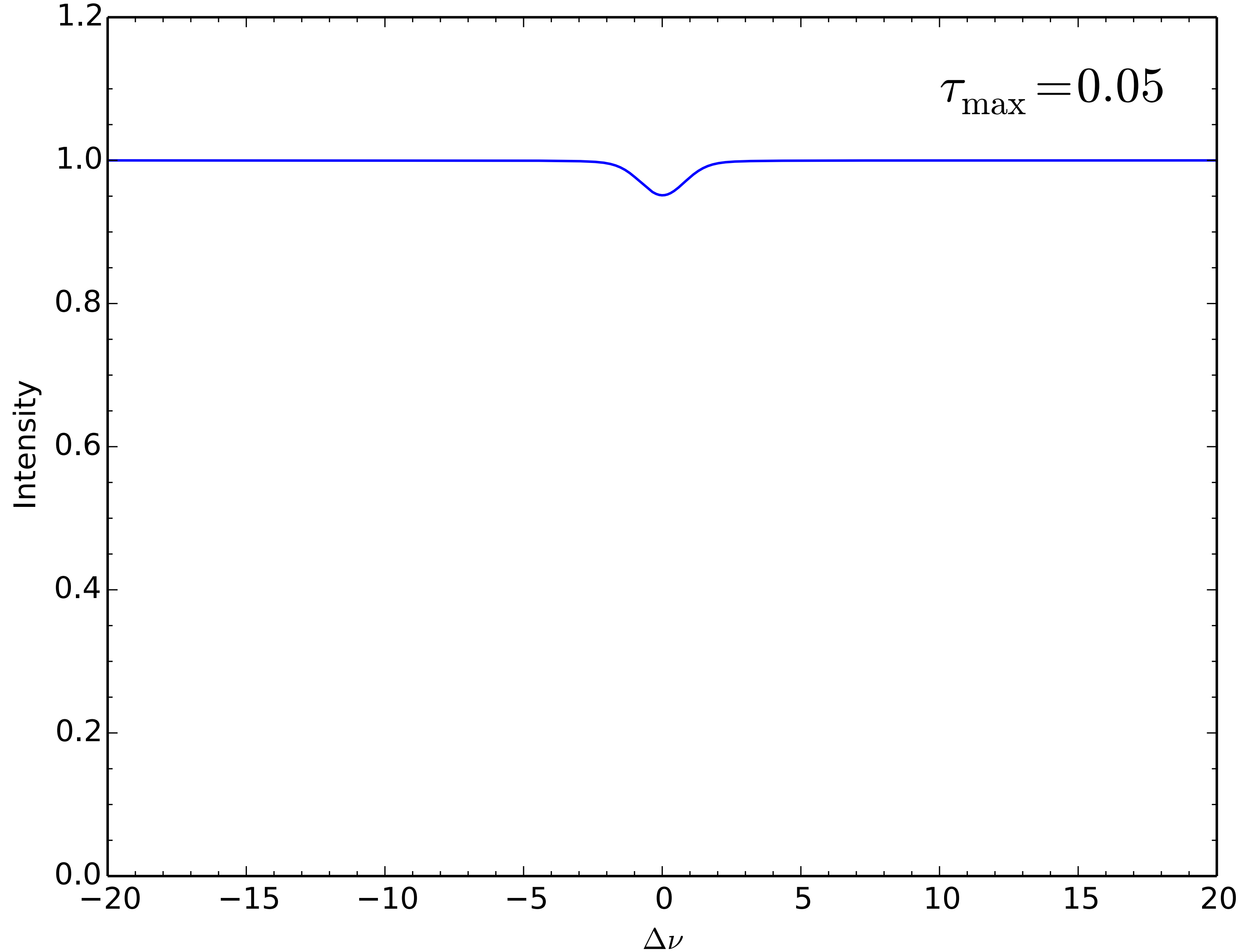
Equivalent width of weak lines

$$W \propto h N_{\mathrm{abs}} f \frac{\pi e^2}{4\pi\epsilon_0 m c}$$

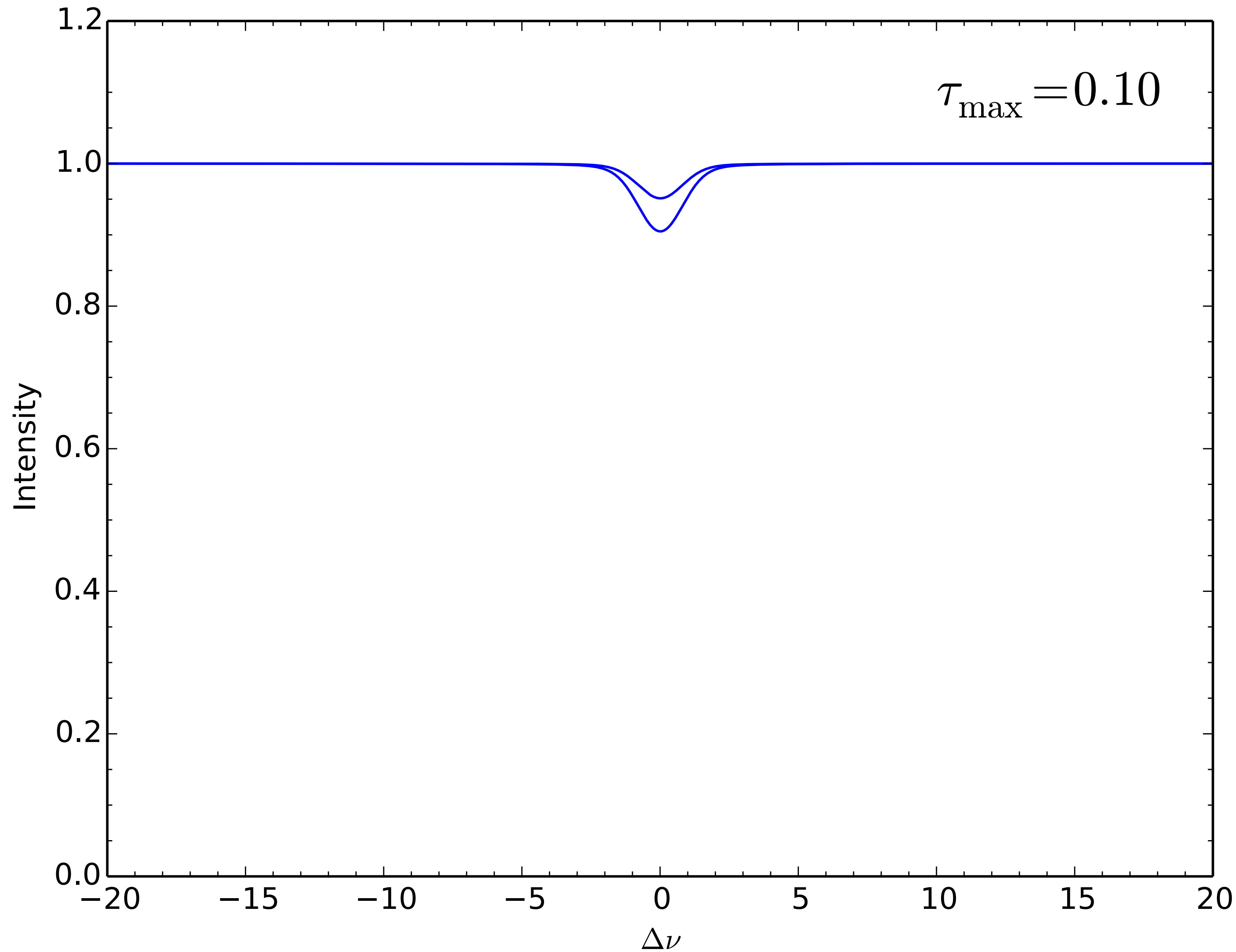
Derived here for a very simplified geometry, but a similar result is true for stellar atmospheres in general.

For *weak* lines, the equivalent width is directly proportional to:
- The number density of absorbers, N_{abs}.
- The oscillator strength, f.

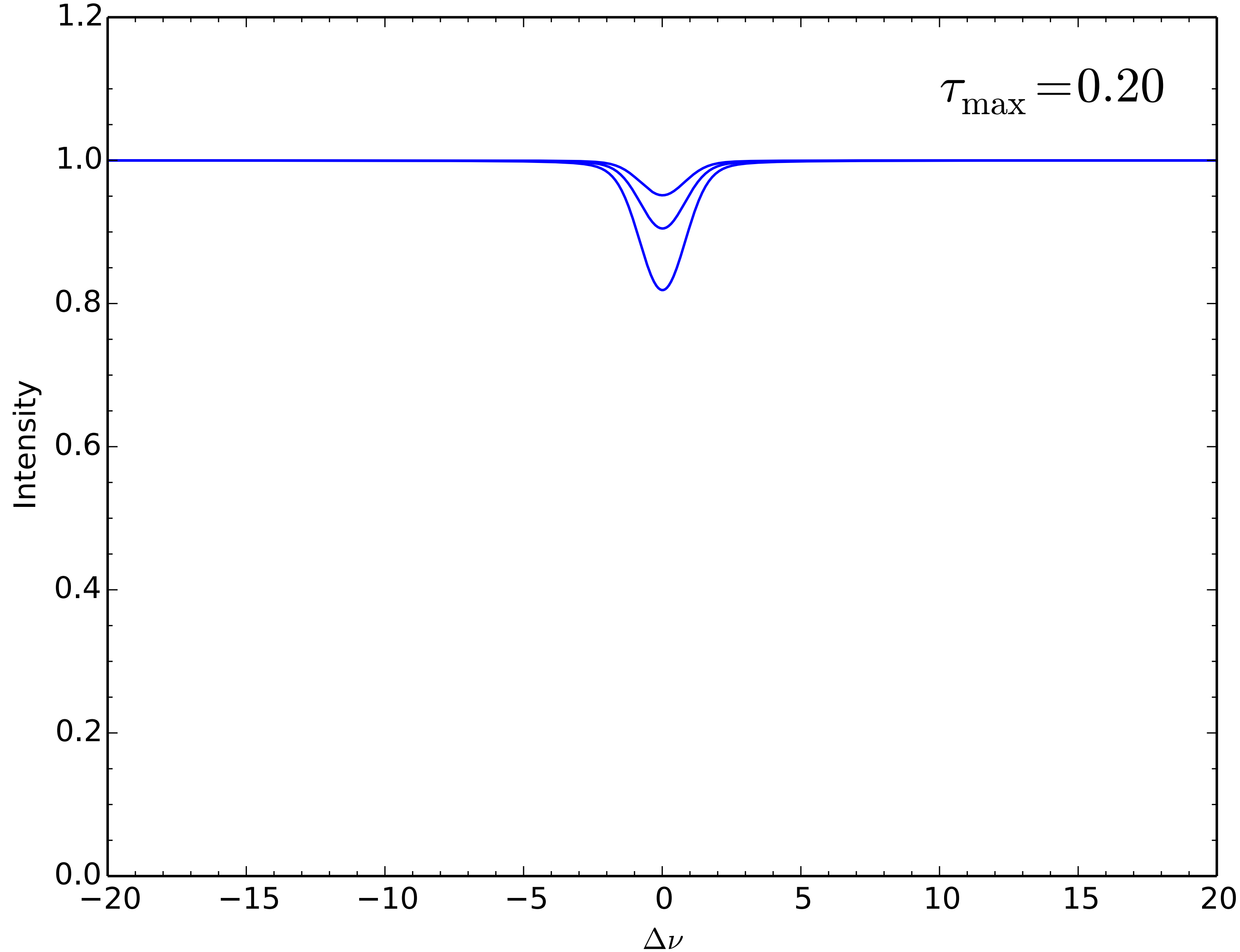
But what about stronger lines?



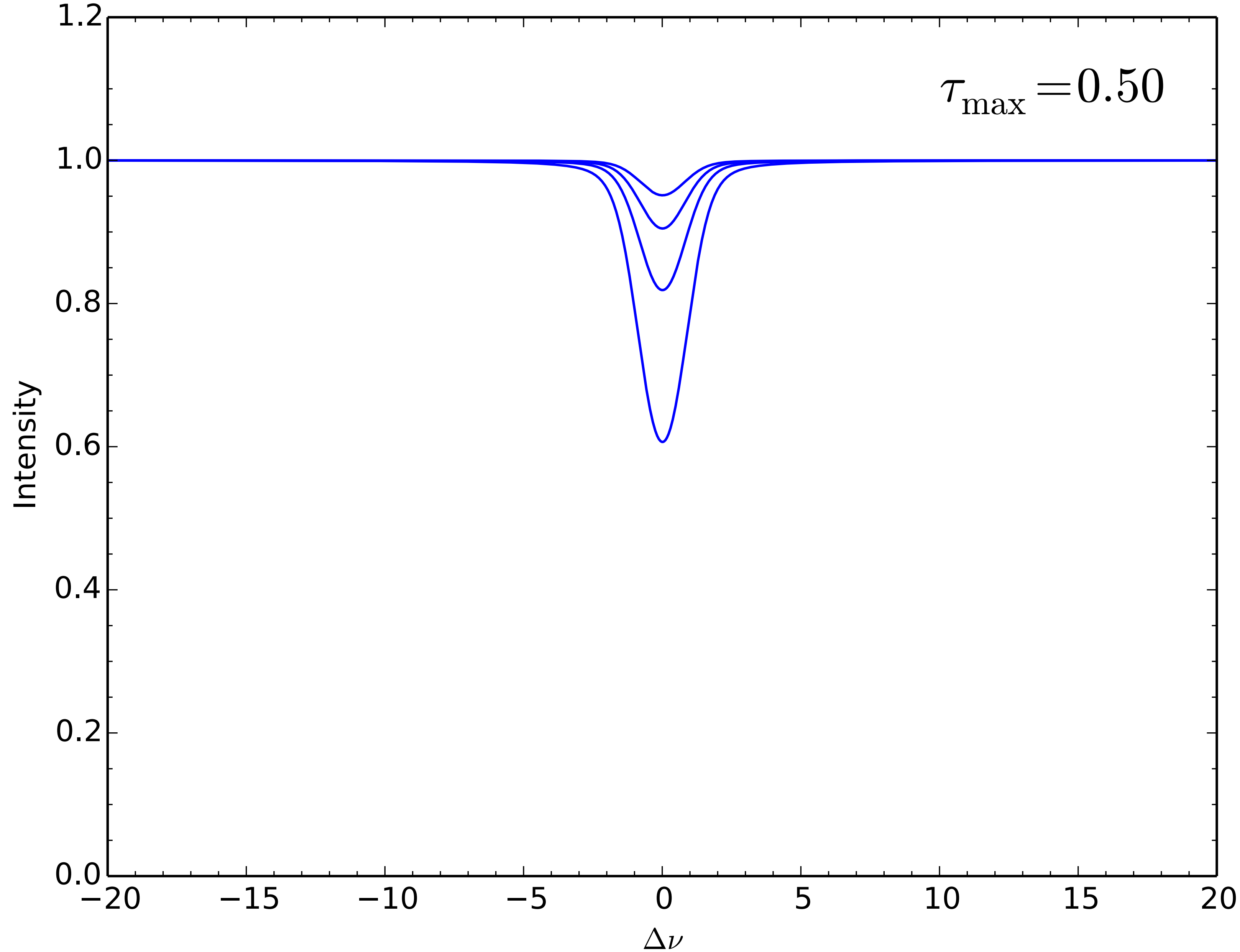
$\tau_{\max} = 0.05$
Intensity
$\Delta\nu$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20



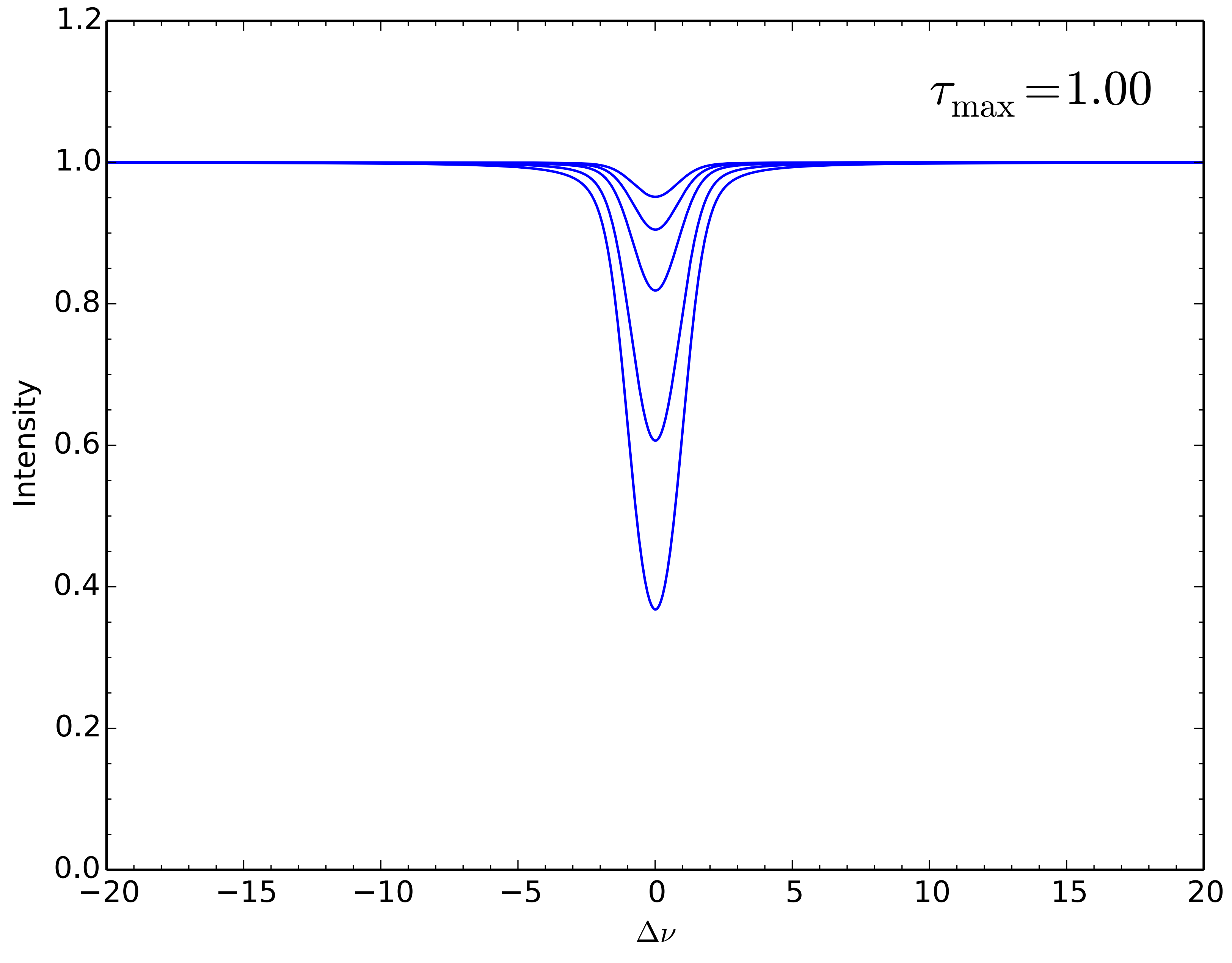
$\tau_{\mathrm{max}} = 0.10$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Intensity
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$

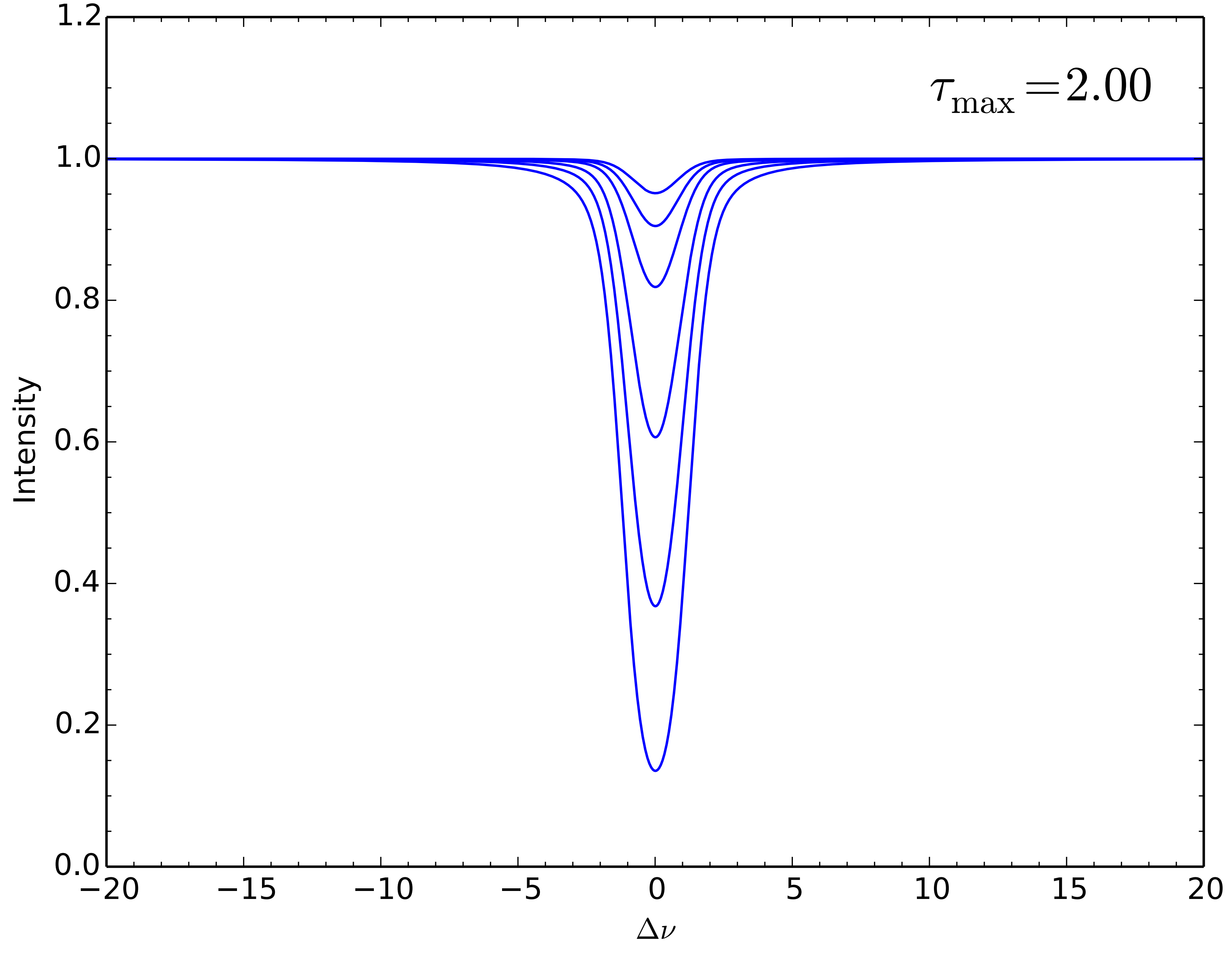


1.2
1.0
0.8
0.6
0.4
0.2
0.0
Intensity
$\tau_{\mathrm{max}} = 0.20$
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$

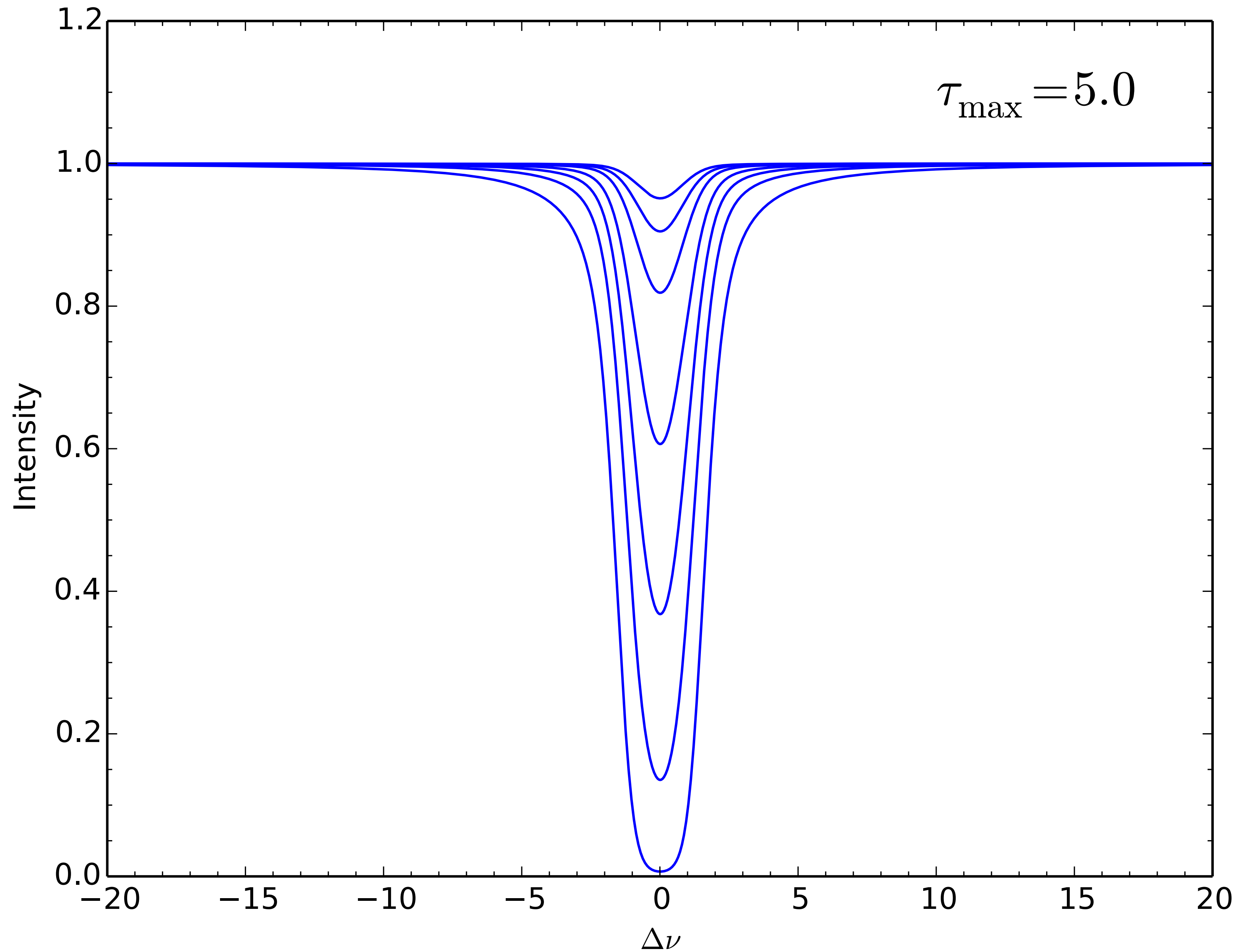


$\tau_{\rm max} = 0.50$
Intensity
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$

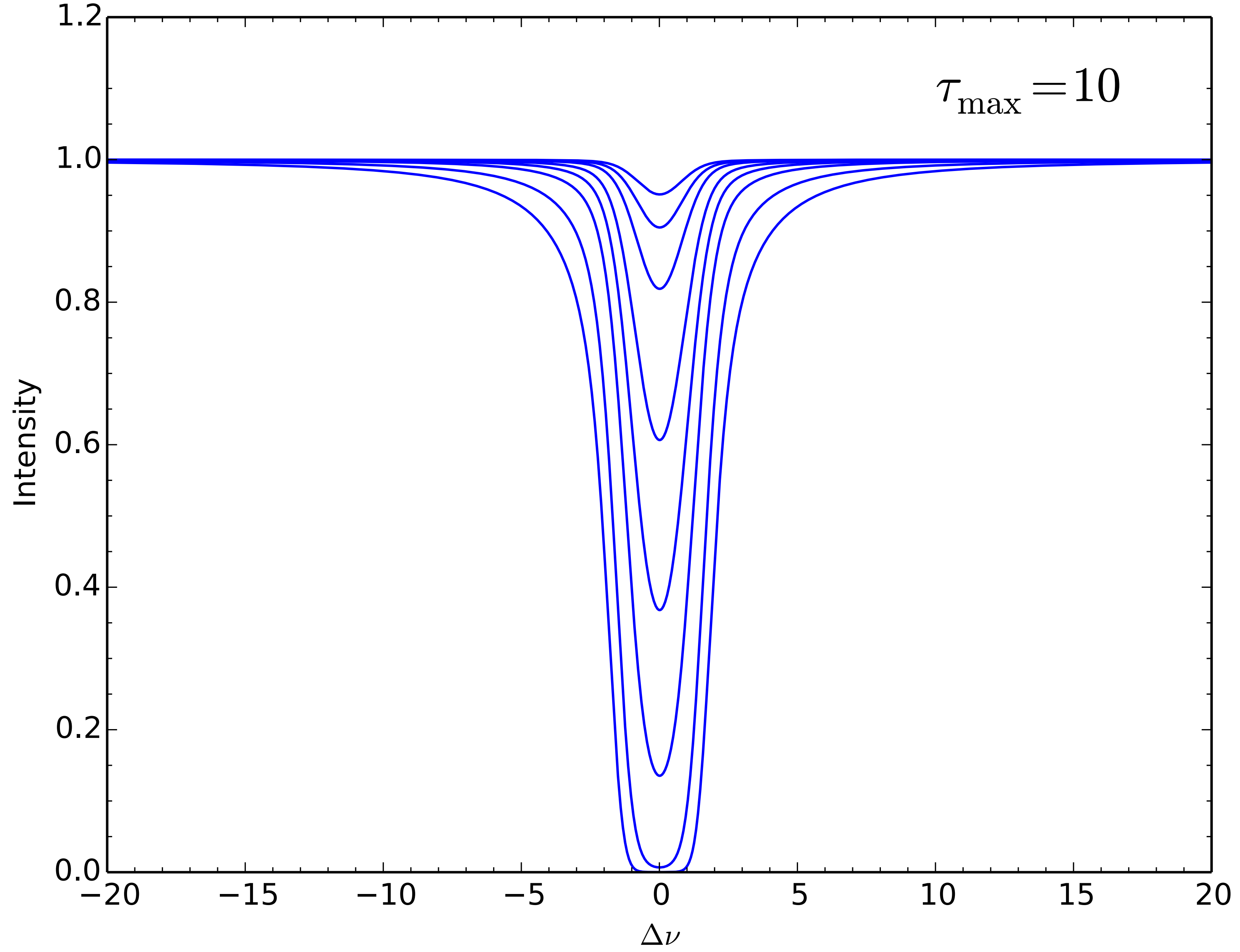

$\tau_{\rm max} = 1.00$
Intensity
$\Delta\nu$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20



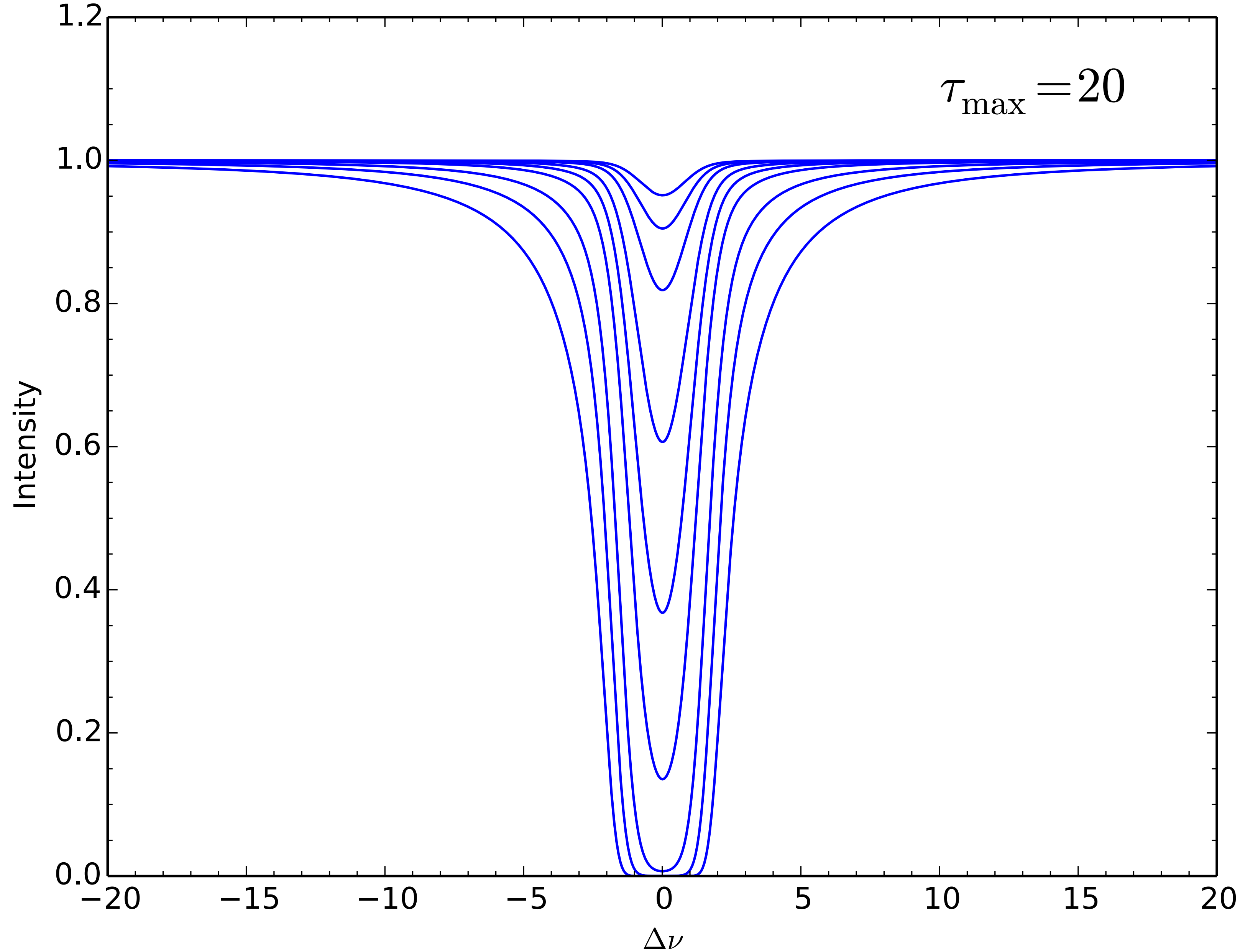
$\tau_{\max} = 2.00$
Intensity
$\Delta\nu$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20



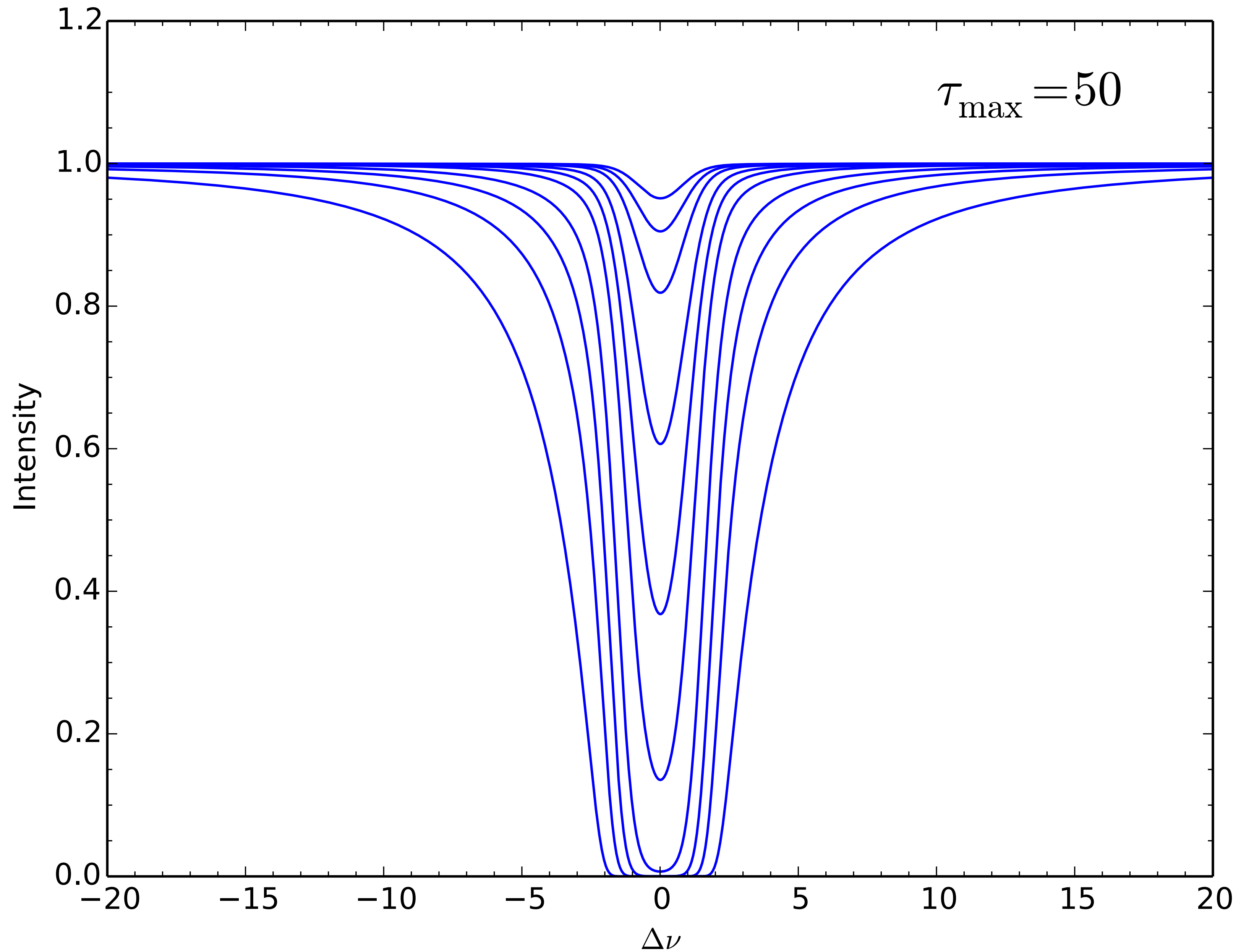
$\tau_{\max} = 5.0$
Intensity
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$



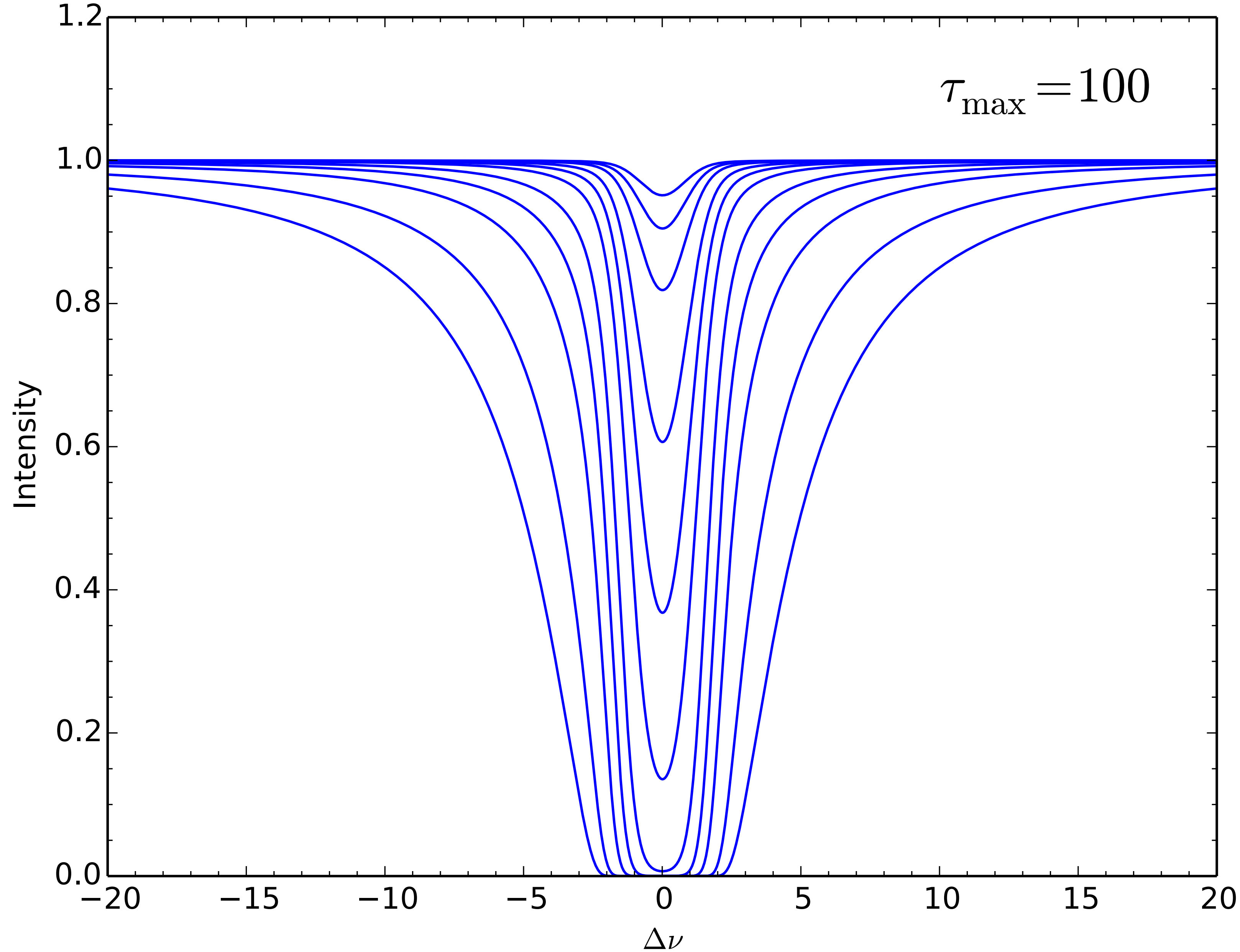
$\tau_{\max} = 10$
Intensity
$\Delta\nu$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20



$\tau_{\max} = 20$
Intensity
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$



$\tau_{\max} = 50$
Intensity
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$


$\tau_{\max} = 100$
Intensity
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−20
−15
−10
−5
0
5
10
15
20
$\Delta\nu$

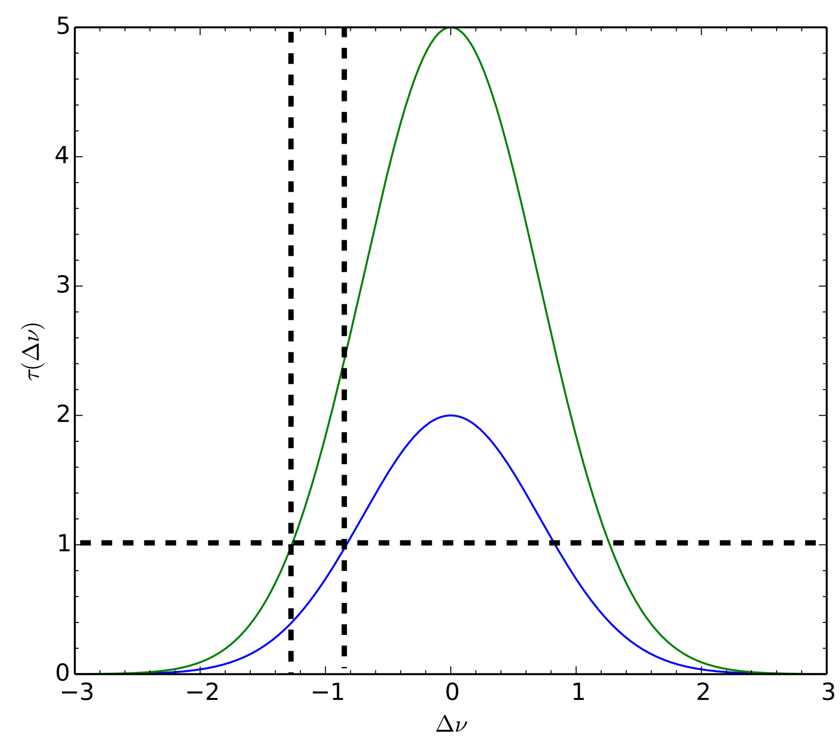
Equivalent width - saturated lines

The equivalent width: $$W = \frac{\lambda^2}{c}\left(1 - \frac{S_1}{I_C}\right)\int (1 - e^{-\tau_\nu})\mathrm{d}\nu$$

The integrand:
$$1 - e^{-\tau_\nu} \approx 0 \;\; \text{for } \tau_\nu \ll 1$$
$$1 - e^{-\tau_\nu} \approx 1 \;\; \text{for } \tau_\nu \gg 1$$

For saturated lines, W depends on the width of the saturated part (τ_ν >> 1):

$$\tau_\nu = \tau_0 e^{-(\Delta\nu/\Delta\nu_D)^2}$$
$$\Delta\nu/\Delta\nu_D = \sqrt{-\log(\tau_\nu/\tau_0)}$$
$$= \sqrt{\log(\tau_0/\tau_\nu)}$$



Hence, the equivalent width scales as

$$W \propto \sqrt{\log(N_{\mathrm{abs}} f)}$$

Equivalent width - damping wings

For strongly saturated lines, the wings of the Lorentz profile eventually dominate:

$$\phi_\nu = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2}$$

In the wings of the line ($(\nu-\nu_0)^2 >> (\mathbf{\Gamma}/4\pi)^2$)

$$\phi_\nu \approx \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2}$$

In this case the “width” of the line scales as

$$W \propto \sqrt{\Gamma N_{\rm abs} f}$$

Note that, in this case, we must know not only f, but also $\mathbf{\Gamma}$ (the damping parameter) to calculate the line profile.

The curve of growth